

# HYPERINVARIANT SUBSPACES FOR SOME $B$ -CIRCULAR OPERATORS

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**ABSTRACT.** We show that if  $A$  is a Hilbert-space operator, then the set of all projections onto hyperinvariant subspaces of  $A$ , which is contained in the von Neumann algebra  $vN(A)$  that is generated by  $A$ , is independent of the representation of  $vN(A)$ , thought of as an abstract  $W^*$ -algebra.

We modify a technique of Foias, Ko, Jung and Pearcy to get a method for finding nontrivial hyperinvariant subspaces of certain operators in finite von Neumann algebras.

We introduce the  $B$ -circular operators as a special case of Speicher's  $B$ -Gaussian operators in free probability theory, and we prove several results about a  $B$ -circular operator  $z$ , including formulas for the  $B$ -valued Cauchy- and  $R$ -transforms of  $z^*z$ . We show that a large class of  $L^\infty([0, 1])$ -circular operators in finite von Neumann algebras have nontrivial hyperinvariant subspaces, and that another large class of them can be embedded in the free group factor  $L(\mathbf{F}_3)$ . These results generalize some of what is known about the quasiniipotent DT-operator.

## 1. INTRODUCTION

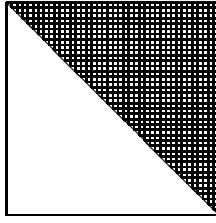
The invariant subspace problem for operators on Hilbert space and the related hyperinvariant subspace problem are both unresolved and are of importance for understanding the structure of Hilbert space operators. Let  $\mathcal{H}$  be a Hilbert space and let  $A \in \mathcal{B}(\mathcal{H})$  be a bounded operator on  $\mathcal{H}$ . A closed subspace  $\mathcal{H}_0 \subseteq \mathcal{H}$  is said to be  $A$ -invariant if  $A(\mathcal{H}_0) \subseteq \mathcal{H}_0$ . (Throughout this paper, all subspaces will be assumed to be closed.) The subspace  $\mathcal{H}_0$  is said to be  $A$ -hyperinvariant if it is  $S$ -invariant whenever  $S \in \mathcal{B}(\mathcal{H})$  commutes with  $A$ . Recall that the invariant subspace problem asks whether, for  $\mathcal{H}$  infinite dimensional, every  $A \in \mathcal{B}(\mathcal{H})$  has an  $A$ -invariant subspace that is nontrivial (i.e. neither  $\{0\}$  nor  $\mathcal{H}$  itself), and the hyperinvariant subspace problem asks whether every  $A \in \mathcal{B}(\mathcal{H})$  that is not a scalar multiple of the identity has a nontrivial  $A$ -hyperinvariant subspace.

Uffe Haagerup [11] made a huge advance on the hyperinvariant subspace problem for operators in  $\text{II}_1$ -factors. He proved that if  $A$  belongs to a  $\text{II}_1$ -factor that is embeddable in the ultrapower  $R^\omega$  of the hyperfinite  $\text{II}_1$ -factor and if the Brown measure [1] of  $A$  is supported on more than one point, then  $A$  has a nontrivial hyperinvariant subspace. (He actually proved much more, namely a result on Brown measure decomposition by restricting to hyperinvariant subspaces.) It is therefore of particular interest to study the hyperinvariant subspace problem for operators whose Brown measure has support reduced to a single point. Since the support of the Brown

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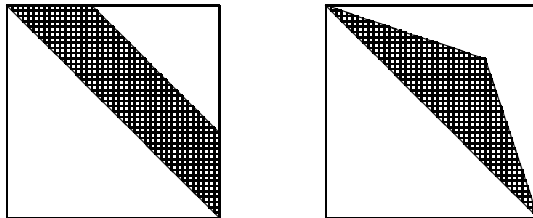
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FIGURE 1. The quasinilpotent DT-operator  $T$ 

measure is contained in the spectrum of the operator, quasinilpotent operators in  $\Pi_1$ -factors are of special interest. The quasinilpotent DT-operator  $T$  in the free group factor  $L(\mathbf{F}_2)$ , from the family of operators defined in [5], was a particularly compelling example to study. The operator  $T$  can be realized as a limit in  $*$ -moments of strictly upper triangular random matrices with i.i.d. complex Gaussian entries above the diagonal. Alternatively, as was seen in [5, §4],  $T$  can be obtained from a semicircular element  $X$  and a free copy of  $L^\infty([0, 1])$  by using projections from the latter to cut out the upper triangular part of  $X$ ; for future reference, note that,  $X$  may be replaced by a circular operator for this procedure. Pictorially, then, we may represent  $T$  as in Figure 1. Here the shaded region has weight 1, the unshaded region has weight 0, and these weights are used to multiply entries of a Gaussian random matrix, as was similarly considered in the self-adjoint case by Shlyakhtenko in [14] and [16].

In [6], Haagerup and the author proved that  $T$  has a one-parameter family of nontrivial hyperinvariant subspaces. The proof utilized precise knowledge of certain  $*$ -moments of  $T$ , conjectured in [5] and proved by Śniady [18], which implies that  $TT^*$  and  $k(T^k(T^*)^k)^{1/k}$  have the same moments for every  $k \in \mathbf{N}$ . It was also shown in [6] that these hyperinvariant subspaces can be characterized in terms of the asymptotic rate of decay of  $\|T^n \xi\|$  as  $n \rightarrow \infty$ , for vectors  $\xi$  in the Hilbert space.

It is natural to consider more general operators than  $T$ , defined also as limits of random matrices or, equivalently, in the approach we will take in this paper, by cutting a circular operator  $Z$  using projections as in [5, §4]. Some of these are pictured in Figure 2, where again the shaded regions indicate weight 1 and the unshaded regions have weight 0. It is natural to ask whether these operators have nontrivial hyperinvariant subspaces. The approach used in [6] for  $T$  is not presently tenable, however; while individual  $*$ -moments for these operators can be calculated rather easily, a good general formula is lacking; moreover, such special relations between

FIGURE 2. Other operators analogous to  $T$

moments of  $TT^*$  and  $T^k(T^*)^k$  as mentioned above are unlikely to be found in more general settings.

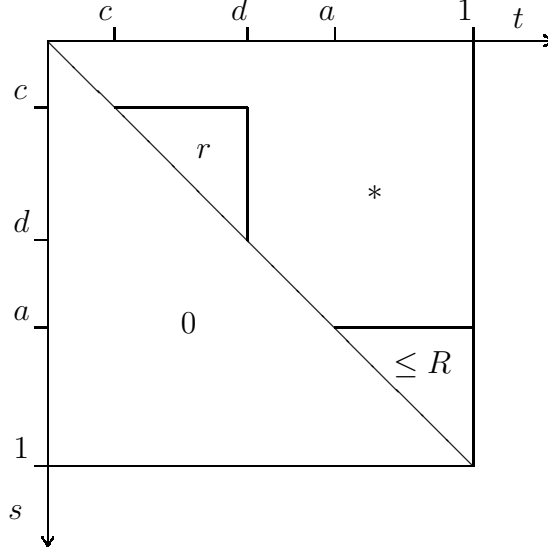
In this paper, we use another technique to exhibit nontrivial hyperinvariant subspaces for all operators in a large class generalizing  $T$ , (including those pictured in Figure 2). This technique is an adaptation of one recently found by Foiaş, Jung, Ko and Percy [10], which they applied to certain quasinilpotent operators  $Q$  in  $\mathcal{B}(\mathcal{H})$ . They consider spectral resolutions of  $Q^k(Q^*)^k$  acting on vectors  $x_0 \in \mathcal{H}$ . Our modification, is, firstly, to take  $Q$  in a  $\text{II}_1$ -factor  $\mathcal{M}$  and for  $x_0$  to take the trace vector in the standard representation of  $\mathcal{M}$ , and, secondly, to consider simultaneously a unital subalgebra  $\mathcal{N} \subseteq \mathcal{M}$  and the conditional expectations of  $Q^k(Q^*)^k$  onto  $\mathcal{N}$  for positive integers  $k$ .

The class of operators we consider are certain  $B$ -circular operators. We introduce  $B$ -circular operators, which are a special case of Speicher's  $B$ -Gaussian operators [19]. Examples include the usual circular operator, Shlyakhtenko's generalized circular operators [15], the quasinilpotent DT-operator  $T$  and the operators pictured in Figure 2. After proving some facts about  $B$ -circular operators, we specialize to  $B$ -circular operators in tracial von Neumann algebras when  $B = L^\infty([0, 1])$ . It turns out that these are the operators  $z_\eta$ , where  $\eta$  is any finite Borel measure on  $[0, 1]^2$  whose push-forwards  $\pi_{i*}\eta$  under the coordinate projections  $\pi_1, \pi_2 : [0, 1]^2 \rightarrow [0, 1]$  are absolutely continuous with respect to Lebesgue measure and have essentially bounded Radon-Nikodym derivatives with respect to Lebesgue measure. When  $\eta$  is Lebesgue measure on  $[0, 1]^2$ , then  $z_\eta$  is the usual circular operator. When  $\eta$  is the restriction of Lebesgue measure to the upper triangle pictured in Figure 1, then  $z_\eta$  is the quasinilpotent DT-operator  $T$ , while when  $\eta$  is, for example, the restriction of Lebesgue measure to one of the shaded regions depicted in Figure 2, then  $z_\eta$  is the corresponding generalization of  $T$  described above. We show that  $z_\eta$  has a nontrivial hyperinvariant subspace whenever the following three criteria hold:

- (i)  $\eta$  is supported in the upper triangle  $\{(s, t) \mid 0 \leq s \leq t \leq 1\}$ ;
- (ii) for some  $0 < c < d$ , the restriction of  $\eta$  to  $\{(s, t) \mid c \leq s \leq t \leq d\}$  is  $r$  times Lebesgue measure, for some  $r > 0$ ;
- (iii) for some  $0 < a < 1$ , the restriction of  $\eta$  to  $\{(s, t) \mid a \leq s \leq t \leq 1\}$  is less than or equal to  $R$  times Lebesgue measure, for some  $R < \infty$ .

These conditions on  $\eta$  are illustrated in Figure 3. (Actually, some weaker conditions on  $\eta$  suffice — see Theorem 5.8 and Figure 4.)

We now describe the contents of the rest of the paper. In §2, we show the well known fact that the projection onto an  $A$ -hyperinvariant subspace belongs to the von Neumann algebra  $vN(A)$  generated by  $A$ . We then show that, given an element  $A$  of a  $W^*$ -algebra  $\mathcal{M}$ , the set of projections in  $\mathcal{M}$  that correspond to  $A$ -hyperinvariant subspaces is independent of the normal  $*$ -representation of  $\mathcal{M}$ . The proof is technically straightforward, but the result is, we believe, conceptually valuable. We also give some related examples. In §3, we prove a version of the construction of hyperinvariant subspaces from [10] applicable to certain operators in a tracial von Neumann algebra. In §4, we introduce  $B$ -circular operators and prove several results about

FIGURE 3. Conditions on  $\eta$ .

them. In §5, we use the method from §3 to construct nontrivial hyperinvariant subspaces for the operators  $z_\eta$  with  $\eta$  satisfying conditions (i)–(iii) above. In §6, we construct  $z_\eta$  in  $L(\mathbf{F}_3)$  when  $\eta$  is absolutely continuous with respect to Lebesgue measure on  $[0, 1]^2$ , using a method analogous to that of [5, §4]. Finally, in §7, we show that  $z_\eta$  is quasiniptotent if  $\eta$  is supported on the upper triangle and is Lebesgue absolutely continuous with bounded Radon–Nikodym derivative near the diagonal.

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## 2. HYPERINVARIANT SUBSPACES OF OPERATORS IN $W^*$ -ALGEBRAS

If  $\mathcal{H}_0$  is a subspace of  $\mathcal{H}$  and if  $p : \mathcal{H} \rightarrow \mathcal{H}_0$  is the projection onto  $\mathcal{H}_0$ , then  $\mathcal{H}_0$  is  $A$ -invariant if and only if  $Ap = pAp$ . (Throughout this paper, all projections will be assumed to be self-adjoint.) We will say that a projection  $p \in \mathcal{B}(\mathcal{H})$  is  $A$ -invariant if  $p\mathcal{H}$  is an  $A$ -invariant subspace.

Let  $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$  be a von Neumann algebra. A subspace  $\mathcal{H}_0 \subseteq \mathcal{H}$  is said to be *affiliated* to  $\mathcal{M}$  if the projection  $p : \mathcal{H} \rightarrow \mathcal{H}_0$  onto  $\mathcal{H}_0$  belongs to  $\mathcal{M}$ . Of particular interest for an operator  $A \in \mathcal{B}(\mathcal{H})$  are  $A$ -invariant subspaces that are affiliated to the von Neumann algebra  $vN(A)$  generated by  $A$ .

The following result is well known and easy to show.

**Proposition 2.1.** *Given a Hilbert space  $\mathcal{H}$  and an operator  $A \in \mathcal{B}(\mathcal{H})$ , if  $\mathcal{H}_0 \subseteq \mathcal{H}$  is an  $A$ -hyperinvariant subspace, then  $\mathcal{H}_0$  is affiliated to the von Neumann algebra  $vN(A)$  generated by  $A$  in  $\mathcal{B}(\mathcal{H})$ .*

*Proof.* Let  $p$  be the projection onto an  $A$ -hyperinvariant subspace. Suppose  $S$  is in the commutant of  $vN(A)$ . Then  $S$  commutes with  $A$ , so  $Sp = pSp$ . But also  $S^*$

commutes with  $A$ , so  $S^*p = pS^*p$  and  $pSp = pS$ . Thus  $pS = Sp$ . By von Neumann's double commutant theorem,  $p \in vN(A)$ .  $\square$

However, there may be  $A$ -invariant subspaces that are affiliated with  $vN(A)$  but are not  $A$ -hyperinvariant, as the following example shows (see also Examples 2.10 and 2.11). Indeed, this is not surprising, because  $vN(A)$  incorporates information about how  $A$  related to its adjoint  $A^*$ , while the (abstract) lattice of hyperinvariant subspaces of  $A$  is a similarity invariant.

*Example 2.2.* Let  $A$  be  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$  acting on a three-dimensional Hilbert space. Then

$$vN(A) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{pmatrix} \middle| a, b_{ij} \in \mathbf{C} \right\}$$

and  $p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is the projection onto a subspace that is  $A$ -invariant and affiliated to  $vN(A)$ , but not  $B$ -invariant, where  $B = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . Note that  $B$  commutes with  $A$ , and thus the range of  $p$  is not  $A$ -hyperinvariant.

It is in any case natural to ask, when  $p \in vN(A)$  is a projection onto an  $A$ -hyperinvariant subspace and when  $\pi : vN(A) \rightarrow \mathcal{B}(\mathcal{K})$  is a normal, faithful  $*$ -homomorphism, whether the range of  $\pi(p)$  in  $\mathcal{K}$  must be a  $\pi(A)$ -hyperinvariant subspace. In other words, given an abstract  $W^*$ -algebra  $\mathcal{M}$  and an element  $A \in \mathcal{M}$ , are the projections onto  $A$ -hyperinvariant subspaces the same for all representations of  $\mathcal{M}$ ?

As is seen below in Theorem 2.5, an affirmative answer to the above question follows readily from the classical result that every normal, faithful  $*$ -homomorphism of a von Neumann algebra is an amplification followed by an induction.

**Lemma 2.3.** *Let  $\mathcal{H}$  and  $\mathcal{K}$  be nonzero Hilbert spaces and let  $\mathcal{H}_0$  be a subspace of  $\mathcal{H}$ . Take  $A \in \mathcal{B}(\mathcal{H})$  and consider the operator  $A \otimes I \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ . Then  $\mathcal{H}_0$  is  $A$ -hyperinvariant if and only if  $\mathcal{H}_0 \otimes \mathcal{K}$  is  $(A \otimes I)$ -hyperinvariant.*

*Proof.* Suppose  $\mathcal{H}_0 \otimes \mathcal{K}$  is  $(A \otimes I)$ -hyperinvariant. If  $\xi \in \mathcal{H}_0$ ,  $S \in \mathcal{B}(\mathcal{H})$  and  $AS = SA$ , then  $(S \otimes I)$  commutes with  $(A \otimes I)$ . Fixing any  $\eta \in \mathcal{K}$ , we have  $(S\xi) \otimes \eta = (S \otimes I)(\xi \otimes \eta) \in \mathcal{H}_0 \otimes \mathcal{K}$  and, therefore,  $S\xi \in \mathcal{H}_0$ . Thus,  $\mathcal{H}_0$  is  $A$ -hyperinvariant.

On the other hand, suppose  $\mathcal{H}_0$  is  $A$ -hyperinvariant. If  $\eta \in \mathcal{K}$ , let  $V_\eta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{K}$  be the map  $V_\eta(\xi) = \xi \otimes \eta$ . Then for every  $B \in \mathcal{B}(\mathcal{H})$ , we have  $V_\eta B = (B \otimes I)V_\eta$ . Suppose  $X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$  commutes with  $A \otimes I$ . Given  $\eta_1, \eta_2 \in \mathcal{K}$ , we have

$$V_{\eta_2}^* X V_{\eta_1} A = V_{\eta_2}^* X (A \otimes I) V_{\eta_1} = A V_{\eta_2}^* X V_{\eta_1},$$

and we deduce  $V_{\eta_2}^* X V_{\eta_1} \mathcal{H}_0 \subseteq \mathcal{H}_0$ . Consequently,  $X(\mathcal{H}_0 \otimes \mathcal{K}) \subseteq \mathcal{H}_0 \otimes \mathcal{K}$ , and  $\mathcal{H}_0 \otimes \mathcal{K}$  is  $(A \otimes I)$ -hyperinvariant.  $\square$

**Lemma 2.4.** *Suppose  $A \in \mathcal{B}(\mathcal{H})$  and  $\mathcal{H}_0 \subseteq \mathcal{H}$  is an  $A$ -hyperinvariant subspace. Let  $P_0 : \mathcal{H} \rightarrow \mathcal{H}_0$  be the projection onto  $\mathcal{H}_0$  and suppose  $E \in \mathcal{B}(\mathcal{H})$  is a projection that commutes with  $A$  and with  $P_0$ . Let  $A_E$  denote the operator in  $\mathcal{B}(E\mathcal{H})$  obtained by restricting  $A$  to  $E\mathcal{H}$ . Then  $E\mathcal{H} \cap \mathcal{H}_0$  is  $A_E$ -hyperinvariant.*

*Proof.* Let  $S \in \mathcal{B}(E\mathcal{H})$  commute with  $A_E$ . Let  $T = SE \in \mathcal{B}(\mathcal{H})$ . Then  $AT = TA$ , so  $T\mathcal{H}_0 \subseteq \mathcal{H}_0$ . But  $T\mathcal{H} \subseteq E\mathcal{H}$ , so  $S(E\mathcal{H} \cap \mathcal{H}_0) = T(\mathcal{H}_0) \subseteq E\mathcal{H} \cap \mathcal{H}_0$ . Therefore,  $E\mathcal{H} \cap \mathcal{H}_0$  is  $A_E$ -hyperinvariant.  $\square$

We let  $\text{Proj}(\mathcal{M})$  denote the set of all projections, i.e. self-adjoint idempotents, in a von Neumann algebra  $\mathcal{M}$ . As promised, the following theorem allows us to speak of hyperinvariant projections of an element of a von Neumann algebra, independent of representation on Hilbert space.

**Theorem 2.5.** *Let  $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$  be a von Neumann algebra, let  $A \in \mathcal{M}$ , let  $p \in \text{Proj}(\mathcal{M})$  and suppose  $p\mathcal{H}$  is a  $A$ -hyperinvariant. Let  $\pi : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$  be any normal, faithful  $*$ -representation of  $\mathcal{M}$ . Then  $\pi(p)(\mathcal{H}_\pi)$  is  $\pi(A)$ -hyperinvariant.*

*Proof.* By [2, Ch. I, §4, Thm. 3], there is a Hilbert space  $\mathcal{K}$ , a projection  $E$  in the commutant of  $\mathcal{M} \otimes I_{\mathcal{K}}$  in  $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$  and a unitary  $U : \mathcal{H}_\pi \rightarrow E(\mathcal{H} \otimes \mathcal{K})$  such that  $\pi(x) = U^*(x \otimes I_{\mathcal{K}})EU$ . But then  $\pi(p)\mathcal{H}_\pi = U^*E((p\mathcal{H}) \otimes \mathcal{K})$ . By Lemma 2.3,  $p\mathcal{H} \otimes \mathcal{K}$  is  $(A \otimes I_{\mathcal{K}})$ -hyperinvariant. From Lemma 2.4, we then obtain that  $E(p\mathcal{H} \otimes \mathcal{K}) = E(\mathcal{H} \otimes \mathcal{K}) \cap (p\mathcal{H} \otimes \mathcal{K})$  is  $E(A \otimes I_{\mathcal{K}})$ -hyperinvariant. Therefore,  $U^*(E(p\mathcal{H} \otimes \mathcal{K})) = \pi(p)\mathcal{H}_\pi$  is  $U^*(E(A \otimes I_{\mathcal{K}}))U$ -hyperinvariant, i.e. is  $\pi(A)$ -hyperinvariant.  $\square$

*Definition 2.6.* Let  $\mathcal{M}$  be a  $W^*$ -algebra, let  $A \in \mathcal{M}$  and let  $p \in \text{Proj}(\mathcal{M})$ . We call  $p$  an  $A$ -hyperinvariant projection if  $\pi(p)\mathcal{H}_\pi$  is a  $\pi(A)$ -hyperinvariant subspace for one (and then for all) normal, faithful  $*$ -homomorphisms  $\pi : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ .

*Remark 2.7.* By a result [3, Cor. 1.5] of Douglas and Pearcy, which utilizes work of Hoover [12], if  $\mathcal{M}$  is a von Neumann algebra that can be written as a direct sum  $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$  with  $\mathcal{M}_1$  a (nonzero) finite type I von Neumann algebra, and if  $A \in \mathcal{M}$  is not a scalar multiple of the identity, then  $A$  has a nontrivial hyperinvariant projection.

In light of Theorem 2.5, it stands to reason that there should be representation-independent descriptions (whatever that may mean) of the  $A$ -hyperinvariant projections in  $vN(A)$ . In that light, it seems natural to ask the following question.

*Question 2.8.* Let  $A$  be an operator in Hilbert space such that the von Neumann algebra  $vN(A)$  it generates is a factor not isomorphic to  $\mathbf{C}$ . If  $p$  is a projection in  $vN(A)$  and if  $p$  is  $S$ -invariant for every element  $S$  of  $vN(A)$  that commutes with  $A$ , is  $p$  necessarily an  $A$ -hyperinvariant projection?

The answer is negative if we do not require  $vN(A)$  be a factor; indeed, the projection  $p$  from Example 2.2 belongs to the center of  $vN(A)$ , but fails to be  $A$ -hyperinvariant. However, as far as the author knows, Question 2.8 is open, (though of course if  $vN(A)$  is a factor of type I, then the answer is positive).

In any case, Examples 2.10 and 2.11 below show that even when  $A$  generates a factor of type I or of type  $\text{II}_1$ , there may be an  $A$ -invariant subspace affiliated to the factor that is not  $A$ -hyperinvariant.

We need a preparatory, elementary lemma about  $n \times n$  matrices. Let  $\{e_{i,j} \mid 1 \leq i, j \leq n\}$  be a system of matrix units in  $M_n(\mathbf{C})$ .

**Lemma 2.9.** *Let  $n, p \in \mathbf{N}$  with  $p \geq 2$ ,  $n > 2p$ . Let  $b_1, \dots, b_{n-p}$  be distinct, strictly positive numbers. Then there is  $\epsilon > 0$  such that whenever  $a_2, \dots, a_p \in (0, \epsilon)$  and*

$$A = \sum_{k=1}^{n-p} b_k e_{k,k+p} + \sum_{k=2}^p a_k e_{1,p+k},$$

*the  $*$ -algebra generated by  $A$  is all of  $M_n(\mathbf{C})$ .*

*Proof.* We may write

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & b_1 & a_2 & \cdots & a_p & 0 & \cdots & 0 \\ & & & & b_2 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & b_p & & & & \\ & & & & & & & b_{p+1} & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & b_{n-p} & \end{pmatrix},$$

where the omitted entries are zero. Let  $\mathfrak{A}$  denote the  $*$ -algebra generated by  $A$ . Let

$$B = \sum_{k=1}^p b_k e_{k,k+p} + \sum_{k=2}^p a_k e_{1,p+k},$$

so that

$$A = B + \sum_{k=p+1}^{n-p} b_k e_{k,k+p}.$$

Then

$$AA^* = BB^* + \sum_{k=p+1}^{n-p} b_k^2 e_{k,k},$$

while

$$BB^* = \sum_{k=1}^p b_k^2 e_{k,k} + \left( \sum_{k=2}^p a_k^2 \right) e_{1,1} + \sum_{k=2}^p b_k a_k (e_{1,k} + e_{k,1}).$$

By choosing  $\epsilon$  sufficiently small, the nonzero eigenvalues of  $BB^*$  can be forced to be arbitrarily close to  $b_1^2, b_2^2, \dots, b_p^2$ , respectively. Then we obtain

$$e_{k,k} \in \mathfrak{A}, \quad (p+1 \leq k \leq n-p) \quad (1)$$

by taking spectral projections of  $AA^*$ . We have  $Ae_{p+1,p+1} = b_1 e_{1,p+1} \in \mathfrak{A}$ , so

$$e_{1,1}, e_{1,p+1} \in \mathfrak{A}. \quad (2)$$

From

$$(A - e_{1,1}A)e_{k+p,k+p} = b_k e_{k,k+p}, \quad (2 \leq k \leq n-p),$$

together with (1) and (2), we get

$$e_{k,k+p} \in \mathfrak{A}, \quad (1 \leq k \leq n-2p). \quad (3)$$

Combined with (1) this yields

$$e_{k,k} \in \mathfrak{A}, \quad (1 \leq k \leq n-p).$$

Using

$$e_{k,k}A = b_k e_{k,k+p}, \quad (2 \leq k \leq n-p),$$

combined with (3), we now get

$$e_{k,k+p} \in \mathfrak{A}, \quad (1 \leq k \leq n-p). \quad (4)$$

From

$$Ae_{k+p,k+p} - b_k e_{k,k+p} = a_k e_{1,k+p}, \quad (2 \leq k \leq p)$$

together with (2), we get

$$e_{1,k+p} \in \mathfrak{A}, \quad (1 \leq k \leq p)$$

and thus also

$$e_{1,k} = e_{1,k+p} e_{k+p,k} \in \mathfrak{A}, \quad (2 \leq k \leq p)$$

and, using (4),

$$e_{1,k+2p} = e_{1,k+p} e_{k+p,k+2p} \in \mathfrak{A}, \quad (1 \leq k \leq p).$$

Continuing as long as possible, we get

$$\begin{aligned} e_{1,k+3p} &= e_{1,k+2p} e_{k+2p,k+3p} \in \mathfrak{A}, & (1 \leq k \leq \min(p, n-3p)), \\ e_{1,k+4p} &= e_{1,k+3p} e_{k+3p,k+4p} \in \mathfrak{A}, & (1 \leq k \leq \min(p, n-4p)), \\ &\vdots \end{aligned}$$

yielding

$$e_{1,j} \in \mathfrak{A}, \quad (1 \leq j \leq n)$$

and  $\mathfrak{A} = M_n(\mathbf{C})$ . □

*Example 2.10.* We will find an operator  $A$ , generating a finite type I factor and having an invariant subspace affiliated to the factor that is not, however,  $A$ -hyperinvariant.

Let  $a > 0$  and consider the upper-triangular  $6 \times 6$  matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & a & & \\ & 0 & 0 & 2 & & \\ & & 0 & 0 & 3 & \\ & & & 0 & 0 & 4 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

where the omitted entries are all zero. By Lemma 2.9, for  $a$  sufficiently small we have  $vN(A) = M_6(\mathbf{C})$ . But taking the Jordan canonical form,  $A$  is similar to

$$B = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 0 & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix},$$

say  $A = SBS^{-1}$ . Thus, the subspace that is the range of the idempotent operator  $S \text{diag}(1, 1, 1, 0, 0, 0) S^{-1}$  is  $A$ -invariant and is affiliated to  $vN(A)$ , but is not  $A$ -hyperinvariant, because it is not invariant under  $SCS^{-1}$ , where  $C = \begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}$ .



*Example 2.11.* We will find an operator  $A$ , generating a type  $\text{II}_1$  factor and having an invariant subspace affiliated to the factor that is not, however,  $A$ -hyperinvariant.

Let  $a > 0$  and consider the upper-triangular  $10 \times 10$  matrix

$$F = \begin{pmatrix} 0 & 0 & 1 & a & a & & & & & \\ & 0 & 0 & 2 & & & & & & \\ & & 0 & 0 & 3 & & & & & \\ & & & 0 & 0 & 4 & & & & \\ & & & & 0 & 0 & 5 & & & \\ & & & & & 0 & 0 & 6 & & \\ & & & & & & 0 & 0 & 7 & \\ & & & & & & & 0 & 0 & 8 \\ & & & & & & & & 0 & 0 \\ & & & & & & & & & 0 \end{pmatrix},$$

where again the omitted entries are all zero. Then

$$F^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 4a & 5a & 6a & & \\ & 0 & 0 & 0 & 0 & 8 & & & & \\ & & 0 & 0 & 0 & 0 & 15 & & & \\ & & & 0 & 0 & 0 & 0 & 24 & & \\ & & & & 0 & 0 & 0 & 0 & 35 & \\ & & & & & 0 & 0 & 0 & 0 & 48 \\ & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 0 \\ & & & & & & & & 0 & 0 \\ & & & & & & & & & 0 \end{pmatrix}.$$

By Lemma 2.9, for sufficiently small  $a$  we have

$$vN(F^2) = M_{10}(\mathbf{C}). \quad (5)$$

Suppose a  $\text{II}_1$ -factor  $\mathcal{M}$  is generated by  $\{u, b\}$ , where  $u$  is a unitary satisfying  $u^2 = 1$  and where  $b \geq 0$  and  $b$  has spectrum in  $[1, 1 + \epsilon]$  for some  $\epsilon > 0$  to be determined later. For example, (see [4]), the interpolated free group factors  $L(\mathbf{F}_t)$  for any  $t \in (1, \frac{3}{2}]$  have generators with these properties. Let  $x = ub$  and consider  $A = F \otimes x \in M_{10}(\mathbf{C}) \otimes \mathcal{M}$ . Let  $F^*F = \sum_{i=1}^n \lambda_i P_i$ , where  $P_1, \dots, P_n$  are orthogonal projections and  $\lambda_1, \dots, \lambda_n$  are the distinct, nonzero eigenvalues of  $F^*F$ . Then

$$A^*A = F^*F \otimes b^2 = \sum_{i=1}^n \lambda_i P_i \otimes b^2.$$

If  $\epsilon$  is small enough, then by taking spectral projections we get  $Q \otimes b^2 \in vN(A)$ , where  $Q = \sum_{i=1}^n P_i$ . Therefore,  $Q \otimes b^{-1} \in vN(A)$  and

$$F \otimes u = A(Q \otimes b^{-1}) \in vN(A).$$

But  $(F \otimes u)^2 = F^2 \otimes 1$ . From (5), we get  $M_{10}(\mathbf{C}) \otimes 1 \subseteq vN(A)$ . Consequently,  $1 \otimes ub \in vN(A)$ , and  $A$  generates the  $\text{II}_1$ -factor  $M_{10}(\mathbf{C}) \otimes \mathcal{M}$ .

However,  $F$  is similar in  $M_{10}(\mathbf{C})$  to its Jordan canonical form

$$G = \begin{pmatrix} 0 & 1 & & & & & & & & \\ & 0 & 1 & & & & & & & \\ & & 0 & 1 & & & & & & \\ & & & 0 & 1 & & & & & \\ & & & & 0 & 0 & & & & \\ & & & & & 0 & 1 & & & \\ & & & & & & 0 & 1 & & \\ & & & & & & & 0 & 1 & \\ & & & & & & & & 0 & 1 \\ & & & & & & & & & 0 \end{pmatrix},$$

so  $A = F \otimes x$  is similar in  $vN(A) = M_{10}(\mathbf{C}) \otimes \mathcal{M}$  to  $G \otimes x$ . Arguing as in Example 2.10, we find a subspace that is  $A$ -invariant but not  $A$ -hyperinvariant and whose projection lies in  $M_{10}(\mathbf{C}) \otimes 1 \subseteq vN(A)$ .

### 3. A CONSTRUCTION OF HYPERINVARIANT SUBSPACES

Foiaş, Jung, Ko and Pearcy [10] recently found a technique that constructs nontrivial hyperinvariant subspaces of some operators on Hilbert space. In this section, we adapt their method so that it will apply to certain operators in tracial von Neumann algebras.

Let  $\mathcal{M}$  be a  $W^*$ -algebra having a normal, faithful, tracial state  $\tau$ . We will consider the singular numbers of operators  $a \in \mathcal{M}$  with respect to  $\tau$ , which were treated by Fack in [8] and by Fack and Kosaki in [9]. Thus, for  $t \in [0, 1]$ , the  $t$ -th singular number of  $a$  is

$$s_t(a) = \inf\{\|a(1-p)\| \mid p \in \text{Proj}(\mathcal{M}), \tau(p) \leq t\}. \quad (6)$$

Of course, the singular numbers are highly dependent on the choice of trace  $\tau$ . We may write  $s_t(a; \tau)$  instead of  $s_t(a)$ , in order to avoid any confusion. By [9, 2.2], we have

$$s_t(a) = \inf\{\lambda \geq 0 \mid \tau(1_{(\lambda, \infty)}(|a|)) \leq t\}, \quad (7)$$

and the infimum is attained. Here,  $1_{(\lambda, \infty)}(|a|)$  denotes the Borel functional calculus, so for  $B \subseteq [0, \infty)$  and  $x \in \mathcal{M}$ ,  $x \geq 0$ ,  $1_B(x)$  denotes the spectral projection for  $x$  corresponding to the set  $B$ .

Let  $\mathcal{M}$  be represented on the Hilbert space  $L^2(\mathcal{M}, \tau)$  via the Gelfand–Naimark–Segal construction. Given  $x \in \mathcal{M}$ , we will let  $\hat{x}$  denote the corresponding element of  $L^2(\mathcal{M}, \tau)$ . Suppose  $\mathcal{N} \subseteq \mathcal{M}$  is a unital  $W^*$ -subalgebra and let  $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$  be the  $\tau$ -preserving conditional expectation onto  $\mathcal{N}$ . As is well known,  $\mathcal{E}$  is obtained by compressing with respect to the projection  $e : L^2(\mathcal{M}, \tau) \rightarrow L^2(\mathcal{N}, \tau|_{\mathcal{N}})$  onto the subspace of  $L^2(\mathcal{M}, \tau)$  that is identified with  $L^2(\mathcal{N}, \tau|_{\mathcal{N}})$  by the inclusion  $\mathcal{N} \subseteq \mathcal{M}$ . In particular

$$exe = e\mathcal{E}(x) = \mathcal{E}(x)e \quad (x \in \mathcal{M}). \quad (8)$$

**Theorem 3.1.** *Let  $b \in \mathcal{M}$ . Suppose there are integers  $p \geq 0$  and  $1 \leq n(1) < n(2) < \dots$  and there are  $\theta \in (0, 1)$  and  $\mu_k \geq 0$ , ( $k \in \mathbf{N}$ ), so that*

$$\lim_{k \rightarrow \infty} \frac{\mu_k}{s_\theta(b^{n(k)})^2} = 0 \quad (9)$$

*and there are vectors  $\zeta_k \in L^2(\mathcal{N}, \tau|_{\mathcal{N}})$  such that*

$$\zeta_k = 1_{[0, \mu_k]}(\mathcal{E}(b^{n(k)+p}(b^*)^{n(k)+p}))\zeta_k \quad (10)$$

*and  $\zeta_k$  converges with respect to the Hilbert-space norm on  $L^2(\mathcal{N}, \tau|_{\mathcal{N}})$  to a nonzero vector  $\zeta \in L^2(\mathcal{N}, \tau|_{\mathcal{N}})$  as  $k \rightarrow \infty$ . Then  $b$  has a nontrivial, hyperinvariant subspace.*

*Proof.* For every  $k \in \mathbf{N}$ , take a sequence  $(c_{k,j})_{j=1}^\infty$  in  $\mathcal{N}$  so that  $\widehat{c_{k,j}}$  converges to  $\zeta_k$  as  $j \rightarrow \infty$ . We may without loss of generality replace  $\mathcal{M}$  by the smallest von Neumann algebra such that  $b \in \mathcal{M}$ , all  $c_{k,j} \in \mathcal{M}$  and  $\mathcal{E}(\mathcal{M}) \subseteq \mathcal{M}$ . Then  $\mathcal{M}$  is countably generated and  $L^2(\mathcal{M}, \tau)$  is separable. Let  $\lambda_k = s_\theta(b^{n(k)})^2 = s_\theta(b^{n(k)}(b^*)^{n(k)})$ . For  $n \in \mathbf{N}$ , let  $E_n$  be the projection-valued spectral measure of  $b^n(b^*)^n$ ; let

$$x_k = \int_{[\lambda_k, \infty)} \frac{1}{t} dE_{n(k)}(t) \in \mathcal{M}$$

and  $y_k = (b^*)^{n(k)}x_k$ . Then  $b^{n(k)}y_k = E_{n(k)}([\lambda_k, \infty))$  and

$$\|b^{n(k)}\widehat{y}_k\|_2^2 = \langle E_{n(k)}([\lambda_k, \infty))^\sim, \hat{1} \rangle = \tau(E_{n(k)}([\lambda_k, \infty))).$$

From (7), we have  $\tau(E_{n(k)}((\lambda, \infty))) > \theta$  whenever  $\lambda < \lambda_k$ , so

$$\theta \leq \inf_{\lambda < \lambda_k} \tau(E_{n(k)}((\lambda, \infty))) = \tau(E_{n(k)}([\lambda_k, \infty))) \leq 1. \quad (11)$$

Since  $\|b^{n(k)}\widehat{y}_k\|_2$  stays bounded as  $k \rightarrow \infty$ , by passing to a subsequence, if necessary, we may without loss of generality assume  $b^{n(k)}\widehat{y}_k$  converges in the weak topology to a vector  $\xi \in L^2(\mathcal{M}, \tau)$  as  $k \rightarrow \infty$ . From (11), we then have  $\langle \xi, \hat{1} \rangle \geq \theta$ , so  $\xi \neq 0$ . Moreover, we have

$$\begin{aligned} \|\widehat{y}_k\|_2^2 &= \tau(x_k b^{n(k)}(b^*)^{n(k)}x_k) \\ &= \tau\left(b^{n(k)}(b^*)^{n(k)} \int_{[\lambda_k, \infty)} \frac{1}{t^2} dE_{n(k)}(t)\right) = \tau(x_k) \leq \frac{1}{\lambda_k}. \end{aligned} \quad (12)$$

Since  $\zeta_k \in L^2(\mathcal{N}, \tau|_{\mathcal{N}})$ , using (8) and (10) we have

$$\begin{aligned} \|(b^*)^{n(k)+p}\zeta_k\|_2^2 &= \langle e b^{n(k)+p}(b^*)^{n(k)+p}e\zeta_k, \zeta_k \rangle = \langle \mathcal{E}(b^{n(k)+p}(b^*)^{n(k)+p})\zeta_k, \zeta_k \rangle \\ &= \langle \mathcal{E}(b^{n(k)+p}(b^*)^{n(k)+p})1_{[0, \mu_k]}(\mathcal{E}(b^{n(k)+p}(b^*)^{n(k)+p}))\zeta_k, \zeta_k \rangle \\ &\leq \mu_k \|\zeta_k\|_2^2. \end{aligned} \quad (13)$$

If  $S \in \mathcal{B}(L^2(\mathcal{M}, \tau))$  and if  $S$  commutes with  $b$ , then we have

$$\langle S\xi, (b^*)^p\zeta \rangle = \lim_{k \rightarrow \infty} \langle S b^{n(k)}\widehat{y}_k, (b^*)^p\zeta_k \rangle = \lim_{k \rightarrow \infty} \langle S\widehat{y}_k, (b^*)^{n(k)+p}\zeta_k \rangle.$$

But from (12) and (13),

$$|\langle S\widehat{y}_k, (b^*)^{n(k)+p}\zeta_k \rangle| \leq \|S\| \|\zeta_k\|_2 \sqrt{\frac{\mu_k}{\lambda_k}}.$$

By hypothesis, this upper bound tends to zero as  $k \rightarrow \infty$ . Therefore, we have

$$\langle S\xi, (b^*)^p\zeta \rangle = 0. \quad (14)$$

Clearly,

$$\mathcal{V} := \overline{\{S\xi \mid S \in \mathcal{B}(L^2(\mathcal{M}, \tau)), Sb = bS\}}$$

is a nonzero  $b$ -hyperinvariant subspace. If  $(b^*)^p\zeta \neq 0$ , then by (14),  $\mathcal{V}$  is nontrivial. If, on the other hand,  $(b^*)^p\zeta = 0$ , then  $b$  has a nonzero cokernel. Since  $b$  is not the zero operator, it follows that  $\overline{b(L^2(\mathcal{M}, \tau))}$  is a nontrivial  $b$ -hyperinvariant subspace.  $\square$

We will make use of the following well known result in application of Theorem 3.1.

**Lemma 3.2.** *Let  $\mathcal{M}$  be a von Neumann with normal, faithful, tracial state  $\tau$ , let  $a \in \mathcal{M}$  and  $q \in \text{Proj}(\mathcal{M})$ . If  $0 < \theta < \tau(q)$ , then*

$$s_\theta(a; \tau) \geq s_{\frac{\theta}{\tau(q)}}(qaq; \tau(q)^{-1}\tau|_{q\mathcal{M}q}). \quad (15)$$

*Proof.* Suppose  $p \in \text{Proj}(\mathcal{M})$ ,  $\tau(p) \leq \theta$ . Then

$$\tau(q \wedge (1 - p)) \geq \tau(q) + \tau(1 - p) - 1 \geq \tau(q) - \theta,$$

so

$$\frac{\tau(q - q \wedge (1 - p))}{\tau(q)} \leq \frac{\theta}{\tau(q)},$$

and

$$\|a(1 - p)\| \geq \|a(q \wedge (1 - p))\| \geq \|qaq(q \wedge (1 - p))\|.$$

This implies (15) directly from the definition (6).  $\square$

#### 4. $B$ -CIRCULAR ELEMENTS

*Definition 4.1.* Let  $B$  be a unital  $*$ -algebra over  $\mathbf{C}$ .

- (i) A  $B$ -valued  $*$ -noncommutative probability space is a pair  $(A, E)$ , where  $A$  is a unital  $*$ -algebra containing  $B$  as a unital  $*$ -subalgebra (which makes  $A$  into a bimodule over  $B$ ) and where  $E : A \rightarrow B$  is a  $B$ -bimodule map satisfying  $E(b) = b$  for all  $b \in B$ .
- (ii) We say  $(A, E)$  is a  $B$ -valued Banach  $*$ -noncommutative probability space if, in addition,  $A$  is a unital Banach  $*$ -algebra,  $B$  is a closed subalgebra of  $A$  and  $E$  is bounded.
- (iii) We say  $(A, E)$  is a  $B$ -valued  $C^*$ -noncommutative probability space if, in addition,  $A$  is a unital  $C^*$ -algebra,  $B$  a  $C^*$ -subalgebra of  $A$  and  $E$  is a projection of norm 1 onto  $B$ . (It follows from [20] that then  $E$  is positive and a  $B$ -bimodule map.)
- (iv) We say  $(A, E)$  is a  $B$ -valued  $W^*$ -noncommutative probability space if, in addition,  $A$  is a unital  $W^*$ -algebra,  $B$  a  $W^*$ -subalgebra of  $A$  and  $E$  is a normal projection of norm 1 onto  $B$ .

*Definition 4.2.* Let  $(A, E)$  be a  $B$ -valued  $*$ -noncommutative probability space. Let  $\alpha : B \rightarrow B$  and  $\beta : B \rightarrow B$  be  $\mathbf{C}$ -linear maps. A  $B$ -circular element with covariance  $(\alpha, \beta)$  is an element  $z \in A$  such that the distribution of the pair  $(z, z^*)$  is  $B$ -Gaussian in the sense of [19, Def. 4.2.3], with covariance determined by

$$\begin{aligned} E(z^*bz) &= \alpha(b) \\ E(zbz^*) &= \beta(b) \\ E(zbz) &= E(z^*bz^*) = E(z) = E(z^*) = 0 \end{aligned}$$

for all  $b \in B$ . In the case that  $(A, E)$  is a  $B$ -valued  $C^*$ -noncommutative probability space, we may call  $z$  a  $B$ -circular operator.

- Examples 4.3.*
- (i) A usual circular operator  $z$  with  $\tau(z^*z) = r$  is, in the notation of Definition 4.2, a  $\mathbf{C}$ -circular element with covariance  $(r, r)$ , where here  $r$  denotes multiplication by  $r$  on  $\mathbf{C}$ .
  - (ii) The generalized circular elements  $\ell_2 + \sqrt{\lambda}\ell_1^*$ ,  $(0 \leq \lambda \leq 1)$ , considered in [15], are  $\mathbf{C}$ -circular with covariance  $(\lambda, 1)$ , where again the scalars indicate operations of multiplication on  $\mathbf{C}$ .

- (iii) A  $\text{DT}(\delta_0, c)$  operator, considered in [5], is  $L^\infty([0, 1])$ -circular, with covariance  $(\alpha, \beta)$ , where

$$\begin{aligned}\alpha(f)(x) &= c^2 \int_0^x f(t) dt \\ \beta(f)(x) &= c^2 \int_x^1 f(t) dt.\end{aligned}$$

This was shown in the appendix to [6].

The  $B$ -valued  $*$ -moments of a  $B$ -circular operator can be calculated using Speicher's free cummulant calculus [19]. This amounts to the nested evaluation described by Śniady in [18, §4.2]. This technique is reviewed below, using the notation  $\pi\{\cdots\}$  for the bracketing of a noncrossing pair partition  $\pi$  with the multiplicative function of free cumulants as in [13].

*Remark 4.4.* With  $z$  a  $B$ -circular operator as above, let  $n \in \mathbf{N}$ ,  $s(1), \dots, s(n) \in \{1, *\}$  and  $b_1, \dots, b_n \in B$ . Then

$$E(z^{s(1)}b_1 z^{s(2)}b_2 \cdots z^{s(n)}b_n) = \sum_{\pi \in \text{NC}_2(n)} \pi\{z^{s(1)}b_1, \dots, z^{s(n)}b_n\} \quad (16)$$

where the sum is over all non-crossing pair partitions  $\pi$  of  $\{1, \dots, n\}$  and where the quantity

$$\pi\{z^{s(1)}b_1, \dots, z^{s(n)}b_n\}, \quad (17)$$

which is the bracketing of the cumulants of the pair  $(z, z^*)$ , is evaluated as described below. In particular, the  $*$ -moment (16) vanishes if  $n$  is odd; so let us assume  $n$  is even. Let

$$\pi = \{\{i_1, j_1\}, \dots, \{i_{n/2}, j_{n/2}\}\}. \quad (18)$$

Then the quantity (17) vanishes unless  $s(i_p) \neq s(j_p)$  for all  $p \in \{1, \dots, n/2\}$ , i.e. unless  $\pi$  pairs only  $z$  with  $z^*$ . Therefore, the  $*$ -moment (16) vanishes if the number of  $j$  such that  $s(j) = *$  differs from the number of  $j$  such that  $s(j) = 1$ . The quantity (17) is evaluated as follows. Without loss of generality take  $i_1 = 1$  in (18). Then

$$\pi\{z^{s(1)}b_1, \dots, z^{s(n)}b_n\} = \begin{cases} \alpha(b_1(\tilde{\pi}'\{z^{s(2)}b_2, \dots, z^{s(j_1-1)}b_{j_1-1}\})) \\ \quad b_{j_1}(\tilde{\pi}''\{z^{s(j_1+1)}b_{j_1+1}, \dots, z^{s(n)}b_n\}) & \text{if } s(1) = *, \\ \beta(b_1(\tilde{\pi}'\{z^{s(2)}b_2, \dots, z^{s(j_1-1)}b_{j_1-1}\})) \\ \quad b_{j_1}(\tilde{\pi}''\{z^{s(j_1+1)}b_{j_1+1}, \dots, z^{s(n)}b_n\}) & \text{if } s(1) = 1, \end{cases} \quad (19)$$

where  $\tilde{\pi}'$  is the restriction of  $\pi$  to  $\{2, \dots, j_1 - 1\}$ , renumbered by left translation to become an element of  $\text{NC}_2(j_1 - 2)$ , while  $\tilde{\pi}''$  is the restriction of  $\pi$  to  $\{j_1 + 1, \dots, n\}$ , renumbered by translation to become an element of  $\text{NC}_2(n - j_1)$ , and where we set  $\tilde{\pi}'\{z^{s(2)}b_2, \dots, z^{s(j_1-1)}b_{j_1-1}\}$  to be 1 if  $j_1 = 2$  and  $\tilde{\pi}''\{z^{s(j_1+1)}b_{j_1+1}, \dots, z^{s(n)}b_n\}$  to be 1 if  $j_1 = n$ .

For example, if  $n = 6$  and  $\pi = \{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$ , then

$$\begin{aligned}\pi\{zb_1, z^*b_2, zb_3, z^*b_4, z^*b_5, zb_6\} &= \beta(b_1(\tilde{\pi}'\{z^*b_2, zb_3\}))b_4\alpha(b_5)b_6 \\ &= \beta(b_1\alpha(b_2)b_3)b_4\alpha(b_5)b_6,\end{aligned}$$

with  $\tilde{\pi}' = \{\{1, 2\}\}$ .

The following basic properties are special instances of Speicher's results [19].

**Proposition 4.5.** *Let  $(A, E)$  be a  $B$ -valued  $*$ -noncommutative probability space and let  $z$  and  $z'$  be  $B$ -circular elements in  $(A, E)$  with covariances  $(\alpha, \beta)$  and  $(\alpha', \beta')$ , respectively. Suppose  $z$  and  $z'$  are  $*$ -free over  $B$  with respect to  $E$ . Then:*

- (i)  $z^*$  is  $B$ -circular with covariance  $(\beta, \alpha)$ .
- (ii)  $z + z'$  is  $B$ -circular with covariance  $(\alpha + \alpha', \beta + \beta')$ .
- (iii) Let  $d \in B$ ; then  $d^*zd$  is  $B$ -circular with covariance  $(\alpha_d, \beta_d)$ , where

$$\begin{aligned}\alpha_d(b) &= d^*\alpha(dbd^*)d \\ \beta_d(b) &= d^*\beta(dbd^*)d.\end{aligned}$$

- (iv) Suppose  $p$  is a self-adjoint idempotent in  $B$ ; then in the  $pBp$ -valued  $*$ -noncommutative probability space  $(pAp, E|_{pAp})$ ,  $pzp$  is  $pBp$ -circular with covariance  $(\tilde{\alpha}_p, \tilde{\beta}_p)$ , where  $\tilde{\alpha}_p, \tilde{\beta}_p : pBp \rightarrow pBp$  are given by

$$\begin{aligned}\tilde{\alpha}_p(b) &= p\alpha(b)p \\ \tilde{\beta}_p(b) &= p\beta(b)p.\end{aligned}$$

*Proof.* Part (i) is clear from the cummulant calculus. Part (ii) follows from the additivity of free cummulants of  $*$ -free variables [19, Thm. 4.1.7]. Part (iii) follows from [19, Prop. 4.1.10]. Part (iv) follows from part (iii).  $\square$

**Proposition 4.6.** *Let  $B$  be a unital  $*$ -algebra, let  $(A, E)$  be a  $B$ -valued  $*$ -probability space, let  $\alpha, \beta : B \rightarrow B$  be  $\mathbf{C}$ -linear maps and let  $z \in A$ . Then  $z$  is a  $B$ -circular operator with covariance  $(\alpha, \beta)$  if and only if*

$$z = \frac{x_1 + ix_2}{\sqrt{2}}, \quad x_i^* = x_i, \quad (20)$$

where in the notation of [19, Def. 4.2.3], the distribution of the pair  $x_1, x_2$  is  $B$ -Gaussian with covariance determined by

$$\begin{aligned}E(x_1bx_1) &= (\alpha(b) + \beta(b))/2 \\ E(x_1bx_2) &= i(\beta(b) - \alpha(b))/2 \\ E(x_2bx_1) &= i(\alpha(b) - \beta(b))/2 \\ E(x_2bx_2) &= (\alpha(b) + \beta(b))/2 \\ E(x_1) &= E(x_2) = 0.\end{aligned}$$

*Proof.* From (20) we have  $x_1 = \frac{z+z^*}{\sqrt{2}}$  and  $x_2 = \frac{z-z^*}{\sqrt{2}}$ . Now the remaining assertions follow from multilinearity of  $B$ -valued cummulants.  $\square$

Compare the following result to [17, Prop. 2.20], from which it follows in light of Proposition 4.6.

**Proposition 4.7.** *Let  $(A, E)$  be a  $B$ -valued  $*$ -noncommutative probability space and let  $z \in A$  be  $B$ -circular with covariance  $(\alpha, \beta)$ . Suppose  $\tau : B \rightarrow B$  is a trace on  $B$ . Then the restriction of  $\tau \circ E$  to the  $*$ -algebra generated by  $B \cup \{z\}$  is a trace if and only if  $\tau(\alpha(b)c) = \tau(\beta(c)b)$  for all  $b, c \in B$ .*

*Notation 4.8.* Let  $(A, E)$  be a  $B$ -valued Banach  $*$ -noncommutative probability space. Given  $x \in A$ , for  $b \in B$  with  $\|b\|$  sufficiently small, we set

$$\tilde{G}_x(b) = \sum_{n=0}^{\infty} E(b(xb)^n).$$

Note that  $\tilde{G}_x$  is related to the Cauchy transform  $G_x(b) = E((b-x)^{-1})$ , as it appears for example in [21] or [22], by  $G_x(b) = \tilde{G}_x(b^{-1})$  for  $b \in B$  invertible with  $\|b^{-1}\|$  sufficiently small.

At the heart of the proof of the following result is a scheme for finding the generating function of the Catalan numbers, when we recall that the Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$  is the number of non-crossing pair partitions of  $\{1, \dots, 2n\}$ . The author is indebted to Lars Aagaard and Uffe Haagerup for discussions of this method for the quasinilpotent DT-operator.

**Proposition 4.9.** *Suppose  $(A, E)$  is a  $B$ -valued Banach  $*$ -noncommutative probability space and  $z \in A$  is a  $B$ -circular operator with covariance  $(\alpha, \beta)$ . Then for  $b, c \in B$  with  $\|b\|\|c\|$  sufficiently small, we have*

$$\tilde{G}_{z^*cz}(b) = b(1 - b\alpha(\tilde{G}_{zbz^*}(c)))^{-1} \quad (21)$$

$$= b(1 - b\alpha(c(1 - c\beta(\tilde{G}_{z^*cz}(b)))^{-1}))^{-1}. \quad (22)$$

Moreover, the  $B$ -valued  $R$ -transform of  $z^*cz$  is given by

$$R_{z^*cz}(b) = \alpha(c(1 - c\beta(b))^{-1}). \quad (23)$$

*Proof.* Using cummulants to evaluate  $*$ -moments of  $z$  as in Remark 4.4, we have for  $n \geq 1$ ,

$$E(b(z^*czb)^n) = \sum_{\text{NC}_2(2n)} b(\pi\{z^*c, zb, \dots, z^*c, zb\}),$$

Any  $\pi \in \text{NC}_2(2n)$  can be uniquely written as

$$\pi = \{\{1, 2k\}\} \cup \pi' \cup \pi'' \quad (24)$$

for some  $k \in \{1, \dots, n\}$ ,  $\pi' \in \text{NC}_2(\{2, \dots, 2k-1\})$  and  $\pi'' \in \text{NC}_2(\{2k+1, \dots, 2n\})$ , where  $\text{NC}_2(S)$  for a subset  $S \subseteq \mathbf{Z}$  is the set of all non-crossing pair partitions of  $S$ , and where we set  $\text{NC}_2(\emptyset) = \{\emptyset\}$ . Moreover, for any  $k \in \{1, \dots, n\}$ , the map

$$(\pi', \pi'') \mapsto \{\{1, 2k\}\} \cup \pi' \cup \pi''$$

is a bijection from  $\text{NC}_2(\{2, \dots, 2k-1\}) \times \text{NC}_2(\{2k+1, \dots, 2n\})$  to  $\{\pi \in \text{NC}_2(2n) \mid \{1, 2k\} \in \pi\}$ . Finally, with  $\pi$  as in (24), we have

$$\pi\{z^*c, zb, \dots, z^*c, zb\} = \alpha(c(\tilde{\pi}'\{zb, z^*c, \dots, zb, z^*c\}))b(\tilde{\pi}''\{z^*c, zb, \dots, z^*c, zb\}),$$

where  $\tilde{\pi}' \in \text{NC}_2(2k-2)$  and  $\tilde{\pi}'' \in \text{NC}_2(2n-2k)$  are obtained from  $\pi'$  and  $\pi''$  by left shifting by 1 and, respectively,  $2k$ . Therefore,

$$\begin{aligned} E(b(z^*czb)^n) &= \sum_{k=1}^n \sum_{\substack{\tilde{\pi}' \in \text{NC}_2(2k-2) \\ \tilde{\pi}'' \in \text{NC}_2(2n-2k)}} b \alpha(c(\tilde{\pi}'\{zb, z^*c, \dots, zb.z^*c\})) \\ &\quad b(\tilde{\pi}''\{z^*c, zb, \dots, z^*c, zb\}) \\ &= \sum_{k=1}^n b \alpha(E(c(zbz^*c)^{k-1})) E(b(z^*czb)^{n-k}). \end{aligned}$$

So we have

$$\begin{aligned} \tilde{G}_{z^*cz}(b) &= \sum_{n=0}^{\infty} E(b(z^*czb)^n) = b + b \sum_{n=1}^{\infty} \sum_{k=1}^n \alpha(E(c(zbz^*c)^{k-1})) E(b(z^*czb)^{n-k}) \\ &= b + b \alpha \left( \sum_{r=0}^{\infty} E(c(zbz^*c)^r) \right) \left( \sum_{s=0}^{\infty} E(b(z^*czb)^s) \right) \\ &= b + b \alpha(\tilde{G}_{zbz^*}(c)) \tilde{G}_{z^*cz}(b). \end{aligned}$$

Solving yields (21). Since  $z^*$  is  $B$ -circular with covariance  $(\beta, \alpha)$ , we have

$$\tilde{G}_{zbz^*}(c) = c(1 - c\beta(\tilde{G}_{z^*cz}(b)))^{-1}.$$

Plugging this into (21) yields (22).

By [19, Thm. 4.1.12], which is due to Voiculescu [21] in a slightly different guise, the  $B$ -valued  $R$ -transform is

$$R_{z^*cz}(b) + b^{-1} = K(b)^{-1},$$

where

$$\tilde{G}_{z^*cz}(K(b)) = K(\tilde{G}_{z^*cz}(b)) = b.$$

Therefore,

$$b = \tilde{G}_{z^*cz}(K(b)) = K(b)(1 - K(b)\alpha(c(1 - c\beta(b))^{-1}))^{-1}.$$

Solving yields

$$K(b)^{-1} = b^{-1} + \alpha(c(1 - c\beta(b))^{-1}),$$

and this gives immediately (23). □

*Remark 4.10.* The formula (22) gives a continued-fraction-type expansion:

$$\tilde{G}_{z^*cz}(b) = \frac{b}{1 - b\alpha\left(\frac{c}{1 - c\beta\left(\frac{b}{1 - b\alpha\left(\frac{c}{1 - \dots}\right)}\right)}\right)}.$$



When  $c = 1$  and when  $b = \zeta^{-1} \in \mathbf{C}$ , we therefore have

$$G_{z^*z}(\zeta) = \tilde{G}_{z^*z}(\zeta^{-1}) = \frac{1}{\zeta - \alpha \left( \frac{1}{\zeta - \beta \left( \frac{1}{\zeta - \alpha \left( \frac{1}{\zeta - \dots} \right)} \right)} \right)}.$$

**Proposition 4.11.** *Let  $B$  be a unital  $C^*$ -algebra and let  $\alpha, \beta : B \rightarrow B$  be  $\mathbf{C}$ -linear maps. Then a  $B$ -circular operator with covariance  $(\alpha, \beta)$  can be realized in a  $B$ -valued  $C^*$ -noncommutative probability space  $(A, E)$  if and only if  $\alpha$  and  $\beta$  are completely positive.*

*Proof.* Necessity follows from the complete positivity of a projection  $E : A \rightarrow B$  onto a  $C^*$ -subalgebra, which was proved by Tomiyama [20]. Sufficiency follows from results of Speicher [19]. Indeed, using Proposition 4.6, complete positivity of  $\alpha$  and  $\beta$  implies that the covariance matrix  $\eta : B \rightarrow M_2(B) = B \otimes M_2(\mathbf{C})$  given by

$$\eta(b) = \frac{1}{2} \begin{pmatrix} \alpha(b) + \beta(b) & i(\beta(b) - \alpha(b)) \\ i(\alpha(b) - \beta(b)) & \alpha(b) + \beta(b) \end{pmatrix} = \alpha(b) \otimes p + \beta(b) \otimes (1 - p),$$

where  $p = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \in M_2(\mathbf{C})$ , is completely positive; by [19, Thm. 4.3.1], the restriction of  $E$  to the  $*$ -algebra  $\mathfrak{A}$  generated by  $\{z\} \cup B$  is positive and by [19, Rmk. 4.3.2], the  $B$ -Gaussian random variables with covariance matrix  $\eta$  can be realized in a  $B$ -valued  $C^*$ -noncommutative probability space.  $\square$

The following exactness result is a direct consequence of Proposition 4.6, [7, Cor. 2.3] and the fact that exactness passes to  $C^*$ -subalgebras.

**Proposition 4.12.** *Let  $B$  be an exact  $C^*$ -algebra, let  $(A, E)$  be a  $B$ -valued  $C^*$ -noncommutative probability space and let  $z \in A$  be a  $B$ -circular element. Then the  $C^*$ -algebra  $C^*(B \cup \{z\})$  is exact.*

**Lemma 4.13.** *Let  $B$  be a unital  $C^*$ -algebra and suppose  $(A, E)$  is a  $B$ -valued  $C^*$ -noncommutative probability space with  $E$  faithful. For  $x \in A$ , we have*

$$\|x\| = \limsup_{n \rightarrow \infty} \|E((x^*x)^n)\|^{1/2n}.$$

*Proof.* We may without loss of generality assume  $A$  and  $B$  are separable. Let  $\phi$  be a faithful state on  $B$ . Then  $\phi \circ E$  is a faithful state on  $A$ . Hence

$$\|x\| = \|x^*x\|^{1/2} = \lim_{n \rightarrow \infty} (\phi \circ E((x^*x)^n))^{1/2n} \leq \limsup_{n \rightarrow \infty} \|E((x^*x)^n)\|^{1/2n} \leq \|x\|.$$

$\square$

**Proposition 4.14.** *Let  $B$  be a unital  $C^*$ -algebra and suppose  $(A, E)$  is a  $B$ -valued  $C^*$ -noncommutative probability space with  $E$  faithful. Let  $\alpha$  and  $\beta$  be completely positive maps from  $B$  to itself. Suppose  $z \in A$  is a  $B$ -circular element with covariance  $(\alpha, \beta)$ . Then*

$$\max(\|\alpha\|, \|\beta\|)^{1/2} \leq \|z\| \leq 2 \max(\|\alpha\|, \|\beta\|)^{1/2}. \quad (25)$$

*Proof.* Let  $K = \max(\|\alpha\|, \|\beta\|)$ . Using the recursive formula (19) for evaluating the bracketing (17), one sees by induction on  $n \geq 1$  that

$$\|\pi\{z^{s(1)}b_1, \dots, z^{s(2n)}b_{2n}\}\| \leq K^n \max(\|b_1\|, \dots, \|b_{2n}\|).$$

From (16), we therefore get

$$\|E(z^{s(1)} \dots z^{s(2n)})\| \leq K^n (\# \text{NC}_2(2n))$$

whenever  $s(1), \dots, s(2n) \in \{1, *\}$ , where  $\# \text{NC}_2(n)$  is the number of non-crossing pair partitions of  $\{1, \dots, n\}$ . Therefore,  $\|E((z^*z)^n)\| \leq K^n \frac{1}{n+1} \binom{2n}{n}$ . Now Lemma 4.13 and the asymptotics of Catalan numbers yield the upper bound in (25).

For the lower bound, we have

$$\begin{aligned} \|z^*z\| &\geq \|E(z^*z)\| = \|\alpha(1)\| = \|\alpha\| \\ \|zz^*\| &\geq \|E(zz^*)\| = \|\beta(1)\| = \|\beta\|. \end{aligned}$$

□

## 5. HYPERINVARIANT SUBSPACES FOR CERTAIN $L^\infty([0, 1])$ -CIRCULAR OPERATORS

For completeness, we provide a proof of the following well known characterization of normal, completely positive maps from  $L^\infty(X, \mu)$  to itself, for  $\mu$  a probability measure. This may be compared to [17, Ex. 2.8], where, however, some conditions are different.

**Lemma 5.1.** *Let  $\mu$  be a probability measure on a measurable space  $(X, \mathcal{M})$ . Let  $\pi_2 : X \times X \rightarrow X$  be the coordinate projection  $\pi_2(x, y) = y$ . Let  $\eta$  be a finite, positive measure on  $(X \times X, \mathcal{M} \otimes \mathcal{M})$  and assume that the push-forward measure  $\pi_{2*}\eta$  is absolutely continuous with respect to  $\mu$  and that the Radon–Nikodym derivative  $\frac{d(\pi_{2*}\eta)}{d\mu}$  is bounded. Then there is a (unique) normal, completely positive map*

$$\alpha_\eta : L^\infty(X, \mu) \rightarrow L^\infty(X, \mu)$$

such that for all  $h \in L^1(X, \mu)$ ,

$$\int_X (\alpha_\eta f)(y) h(y) d\mu(y) = \int_{X \times X} f(x) h(y) d\eta(x, y). \quad (26)$$

We may formally write

$$(\alpha_\eta f)(y) = \int_X f(x) \eta(dx, y).$$

Conversely, every normal, completely positive map

$$\alpha : L^\infty(X, \mu) \rightarrow L^\infty(X, \mu) \quad (27)$$

arises in this way from a measure  $\eta$ , and

$$\frac{d(\pi_{2*}\eta)}{d\mu} = \alpha(1_X). \quad (28)$$

Consequently,

$$\|\alpha\| = \|\alpha(1_X)\|_\infty = \left\| \frac{d(\pi_{2*}\eta)}{d\mu} \right\|_\infty. \quad (29)$$

*Proof.* We have

$$\begin{aligned} \int |f(x)h(y)|d\eta(x, y) &\leq \|f\|_\infty \int |h(y)|d\eta(x, y) = \|f\|_\infty \int |h(y)|d(\pi_{2*}\eta)(y) \\ &\leq \|f\|_\infty \left\| \frac{d(\pi_{2*}\eta)}{d\mu} \right\|_\infty \int |h(y)|d\mu(y) = \|f\|_\infty \left\| \frac{d(\pi_{2*}\eta)}{d\mu} \right\|_\infty \|h\|_{L^1(\mu)}. \end{aligned}$$

So (26) uniquely defines an element  $\alpha_\eta f$  of  $L^\infty(X, \mu)$ . Clearly the map  $\alpha_\eta$  is positive (therefore, completely positive) and normal.

Conversely, given a normal, completely positive map  $\alpha$  as in (27), for  $E_1, E_2 \in \mathcal{M}$  define

$$\eta(E_1 \times E_2) = \int_{E_2} (\alpha(1_{E_1}))(y)d\mu(y). \quad (30)$$

Using positivity and normality,  $\eta$  is seen to extend to a finite, positive measure on  $X \times X$ . Finally,  $\pi_{2*}\eta(E) = \eta(X \times E)$  and from (30) we get that  $\pi_{2*}\eta$  is  $\mu$ -absolutely continuous and (28) holds. Now equation (29) follows directly.  $\square$

If we desire a completely positive map  $\beta_\eta : L^\infty(X, \mu) \rightarrow L^\infty(X, \mu)$  such that

$$\tau(\alpha_\eta(f)g) = \tau(f\beta_\eta(g))$$

for all  $f, g \in L^\infty(X, \mu)$ , where  $\tau(\cdot) = \int \cdot d\mu$ , (cf. Proposition 4.7), then  $\beta_\eta$  will need to satisfy

$$\int_X (\beta_\eta g)(x)h(x)d\mu(x) = \int_{X \times X} h(x)g(y)d\eta(x, y)$$

for all  $h \in L^1(X, \mu)$ , and we will need also  $\pi_{1*}\eta$  to be absolutely continuous with respect to  $\mu$  and have bounded Radon–Nikodym derivative, where  $\pi_1 : X \times X \rightarrow X$  is the other coordinate projection. We may formally write

$$(\beta_\eta g)(x) = \int_X g(y)\eta(x, dy).$$

Consider  $\mathcal{D} = L^\infty([0, 1])$  with trace  $\tau$  given by integration with respect to Lebesgue measure. We will study  $\mathcal{D}$ -circular operators in a  $\mathcal{D}$ -valued  $W^*$ -noncommutative probability space  $(\mathcal{M}, \mathcal{E})$  such that  $\tau \circ \mathcal{E}$  is a normal, faithful, tracial state on  $\mathcal{M}$ . In light of the above discussion, this class of operators is precisely the class delineated below.

*Definition 5.2.* If  $\eta$  is a finite, Borel measure on  $[0, 1]^2$  whose push-forward measures  $\pi_{i*}\eta$  under both coordinate projections  $\pi_1, \pi_2 : [0, 1]^2 \rightarrow [0, 1]$  are absolutely continuous with respect to Lebesgue measure and have bounded Radon–Nikodym derivative,

let  $z_\eta$  be a  $\mathcal{D}$ -circular operator with covariance  $(\alpha_\eta, \beta_\eta)$ , where, for all  $h \in L^1([0, 1])$ ,

$$\begin{aligned} \int_0^1 (\alpha_\eta f)(y) h(y) dy &= \int_{[0,1]^2} f(x) h(y) d\eta(x, y), \\ \int_0^1 (\beta_\eta g)(x) h(x) dx &= \int_{[0,1]^2} h(x) g(y) d\eta(x, y). \end{aligned} \quad (31)$$

It follows from Propositions 4.11 and 4.7 that such a  $\mathcal{D}$ -circular operator  $z_\eta$  exists in a tracial  $\mathcal{D}$ -valued  $C^*$ -noncommutative probability space, and the Gelfand–Naimark–Segal construction then yields  $z_\eta$  in a  $\mathcal{D}$ -valued  $W^*$ -noncommutative probability space  $(\mathcal{M}, \mathcal{E})$ , with  $\tau \circ \mathcal{E}$  a normal, faithful, tracial state on  $\mathcal{M}$ . We may also use  $\tau$  to denote the faithful trace  $\tau \circ \mathcal{E}$  on  $\mathcal{M}$ , and for  $a \in \mathcal{M}$ , we let  $\|a\|_2 = \tau(a^*a)^{1/2}$ , as usual.

**Lemma 5.3.**  $\|z_\eta\|_2 = \eta([0, 1]^2)^{1/2}$ .

*Proof.* We have

$$\|z_\eta\|_2^2 = \tau(z_\eta^* z_\eta) = \tau \circ \mathcal{E}(z_\eta^* z_\eta) = \tau(\alpha_\eta(1)) = \int_0^1 (\alpha_\eta(1))(y) dy = \eta([0, 1]^2),$$

where the last equality is from (31).  $\square$

*Notation 5.4.* For Borel subsets  $A$  of  $\mathbf{R}$ , we will use  $1_A$  for the characteristic function of  $A$ , for example as in  $1_A(S)$ , when applied via the Borel functional calculus to a self-adjoint operator  $S \in \mathcal{M}$ . On the other hand, the notation  $\chi_A$  for  $A \subseteq [0, 1]$  will be used for the characteristic function of  $A$  considered as an element of  $\mathcal{D} = L^\infty([0, 1]) \subseteq \mathcal{M}$ .

**Lemma 5.5.** *If  $A$  and  $B$  are Borel subsets of  $[0, 1]$  and if  $\eta(A \times B) = 0$ , then  $\chi_A z_\eta \chi_B = 0$ .*

*Proof.* We have

$$\begin{aligned} \tau \circ \mathcal{E}((\chi_A z_\eta \chi_B)^* (\chi_A z_\eta \chi_B)) &= \tau(\chi_B \mathcal{E}(z_\eta^* \chi_A z_\eta)) = \tau(\chi_B \alpha_\eta(\chi_A)) \\ &= \int_{[0,1]^2} \chi_A(x) \chi_B(y) d\eta(x, y) = \eta(A \times B) = 0. \end{aligned}$$

$\square$

**Lemma 5.6.** *Suppose for some  $0 \leq a < 1$ , the restriction of  $\eta$  to  $[a, 1] \times [0, 1]$  is absolutely continuous with respect to Lebesgue measure, has bounded Radon–Nikodym derivative and is supported in  $\{(s, t) \mid a \leq s \leq t \leq 1\}$ . Then there is  $K > 0$  such that for all  $n \in \mathbf{N}$  and all  $\mu \in [0, \frac{K^n(1-a)^n}{n!}]$ , we have*

$$1_{[0, \mu]}(\mathcal{E}(z_\eta^n (z_\eta^*)^n)) \geq \chi_{[\rho, 1]},$$

where

$$\rho = 1 - \frac{(n! \mu)^{1/n}}{K}. \quad (32)$$

Consequently, if we fix  $\gamma \in [a, 1)$ , then letting  $\zeta = (\chi_{[\gamma, 1]})^\wedge \in L^2(\mathcal{D}, \tau|_{\mathcal{D}})$  and letting

$$\mu_n = \frac{K^n(1 - \gamma)^n}{n!}, \quad (33)$$

we have

$$1_{[0, \mu_n]}(\mathcal{E}(z_\eta^n(z_\eta^*)^n))\zeta = \zeta$$

for all  $n \in \mathbf{N}$ .

*Proof.* Let  $H$  be the Radon–Nikodym derivative of the restriction of  $\eta$  to  $[a, 1] \times [0, 1]$  with respect to Lebesgue measure and let  $K > 0$  be at least as large as the essential supremum  $\|H\|_\infty$  of  $H$ . Since  $\eta([a, 1] \times [0, a]) = 0$ , from Lemma 5.5 we have  $\chi_{[a, 1]}z_\eta = \chi_{[a, 1]}z_\eta\chi_{[a, 1]}$  and consequently,

$$\mathcal{E}(z_\eta^n(z_\eta^*)^n) \geq \chi_{[a, 1]}\mathcal{E}(z_\eta^n(z_\eta^*)^n) = \mathcal{E}((\chi_{[a, 1]}z_\eta)^n(\chi_{[a, 1]}^*z_\eta^*)^n). \quad (34)$$

Let  $f_n$  denote the right-hand-side of (34), with  $f_0 = \chi_{[a, 1]}$ . Then by the nested evaluation described in Remark 4.4, we get  $f_{n+1} = \chi_{[a, 1]}\beta_\eta(f_n)$ , for all  $n \geq 0$ . We therefore have, whenever  $a \leq x \leq 1$ ,

$$0 \leq f_{n+1}(x) = (\beta_\eta f_n)(x) = \int_a^1 f_n(y)H(x, y)dy \leq K \int_x^1 f_n(y)dy.$$

It follows by induction on  $n \geq 0$  that

$$0 \leq f_n(x) \leq K^n \frac{(1 - x)^n}{n!}, \quad (a \leq x \leq 1).$$

If  $0 \leq \mu \leq \frac{K^n(1-a)^n}{n!}$ , then

$$1_{[0, \mu]}(\mathcal{E}(z_\eta^n(z_\eta^*)^n)) \geq 1_{[0, \mu]}(f_n) \geq 1_{[0, \mu]}(\frac{K^n(1-x)^n}{n!}) = \chi_{[\rho, 1]},$$

where  $\frac{K^n(1-\rho)^n}{n!} = \mu$ , i.e. where (32) holds. The remaining assertions follow directly.  $\square$

**Lemma 5.7.** *Let  $0 \leq c < d \leq 1$  and suppose*

$$\eta([d, 1] \times [c, d]) = 0 = \eta([c, 1] \times [0, c]). \quad (35)$$

*Let  $\phi : [c, d] \rightarrow [0, 1]$  be  $\phi(x) = (x - c)/(d - c)$ . Let*

$$\eta' = (d - c)^{-1}(\phi \times \phi)_*(\eta|_{[c, d]^2})$$

*be the measure on  $[0, 1]^2$  that is  $(d - c)^{-1}$  times the push-forward under  $\phi \times \phi$  of the restriction of  $\eta$  to  $[c, d] \times [c, d]$ . Then whenever  $0 \leq \theta \leq d - c$  and  $n \in \mathbf{N}$ , we have*

$$s_\theta(z_\eta^n) \geq s_{\frac{\theta}{d-c}}(z_{\eta'}^n). \quad (36)$$

*Proof.* Let  $p_1 = \chi_{[0, c]}$  and  $p_2 = \chi_{[c, d]}$ . From (35) and Lemma 5.5,  $z_\eta p_2 = (p_1 + p_2)z_\eta p_2$  and  $z_\eta p_1 = p_1 z_\eta p_1$ , so  $p_2 z_\eta^n p_2 = (p_2 z_\eta p_2)^n$ . Consequently, by Lemma 3.2,

$$s_\theta(z_\eta^n) \geq s_{\frac{\theta}{d-c}}((p_2 z_\eta p_2)^n; (d - c)^{-1}(\tau \circ \mathcal{E})|_{p_2 \mathcal{M}_{p_2}}). \quad (37)$$

By Proposition 4.5(iv),  $p_2 z_\eta p_2$  is a  $p_2 \mathcal{D}$ -circular element in  $(p_2 \mathcal{M}_{p_2}, \mathcal{E}|_{p_2 \mathcal{M}_{p_2}})$  with covariance  $(\tilde{\alpha}, \tilde{\beta})$ , where for  $f \in p_2 \mathcal{D} = L^\infty([c, d])$ ,  $\tilde{\alpha}(f) = p_2 \alpha_\eta(f)$  and  $\tilde{\beta}(f) = p_2 \beta_\eta(f)$ .

Consider the isomorphism  $\tilde{\phi} : L^\infty([0, 1]) \rightarrow L^\infty([c, d])$  given by  $\tilde{\phi}(f) = f \circ \phi$ . Using this identification, we may regard  $(p_2 \mathcal{M}_{p_2}, \mathcal{E}|_{p_2 \mathcal{M}_{p_2}})$  as an  $L^\infty([0, 1])$ -valued  $W^*$ -noncommutative probability space and  $p_2 z_\eta p_2$  as an  $L^\infty([0, 1])$ -circular operator with covariance  $(\tilde{\phi}^{-1} \circ \tilde{\alpha} \circ \tilde{\phi}, \tilde{\phi}^{-1} \circ \tilde{\beta} \circ \tilde{\phi})$ . Let  $f \in L^\infty([0, 1])$  and  $h \in L^1([0, 1])$ . Then

$$\begin{aligned} \int_0^1 ((\tilde{\phi}^{-1} \circ \tilde{\alpha} \circ \tilde{\phi})f)(t)h(t)dt &= \int_0^1 ((\tilde{\alpha} \circ \tilde{\phi})f)(c + (d - c)t)h(t)dt \\ &= (d - c)^{-1} \int_c^d ((\tilde{\alpha} \circ \tilde{\phi})f)(y)h\left(\frac{y - c}{d - c}\right)dy \\ &= (d - c)^{-1} \int_{[c, d]^2} (\tilde{\phi}f)(x)h\left(\frac{y - c}{d - c}\right)d\eta(x, y) \\ &= (d - c)^{-1} \int_{[c, d]^2} (f \circ \phi)(x)(h \circ \phi)(y)d\eta(x, y) \\ &= (d - c)^{-1} \int_{[0, 1]^2} f(s)h(t)d((\phi \times \phi)_*\eta)(s, t) \\ &= \int_0^1 (\alpha_{\eta'}f)(t)h(t)dt. \end{aligned}$$

Thus,  $\tilde{\phi}^{-1} \circ \tilde{\alpha} \circ \tilde{\phi} = \alpha_{\eta'}$ . Similarly, we have  $\tilde{\phi}^{-1} \circ \tilde{\beta} \circ \tilde{\phi} = \beta_{\eta'}$ . Hence,  $p_2 z_\eta p_2$  is identified with  $z_{\eta'}$ . Finally, note that the trace  $(d - c)^{-1} \tau \circ \mathcal{E}|_{p_2 \mathcal{M}_{p_2}}$  is equal to  $\tau \circ \tilde{\phi}^{-1} \circ \mathcal{E}|_{p_2 \mathcal{M}_{p_2}}$ , and (36) follows from (37).  $\square$

**Theorem 5.8.** *Consider an  $L^\infty([0, 1])$ -circular operator  $z_\eta$  as described in Definition 5.2. Suppose*

- (i) *for some  $0 \leq a < 1$ , the restriction of  $\eta$  to  $\{(s, t) \mid a \leq s \leq t \leq 1\}$  is less than or equal to  $R$  times Lebesgue measure, for some  $R < \infty$ ;*
- (ii) *for some  $0 \leq c < d \leq 1$ , the restriction of  $\eta$  to  $\{(s, t) \mid c \leq s \leq t \leq d\}$  is  $r$  times Lebesgue measure for some  $r > 0$ ;*
- (iii)  *$\eta$  vanishes on*

$$\begin{aligned} &([c, 1] \times [0, c]) \cup ([d, 1] \times [c, d]) \cup ([a, 1] \times [0, 1]) \\ &\cup \{(s, t) \mid c \leq t \leq s \leq d\} \cup \{(s, t) \mid a \leq t \leq s \leq a\}. \end{aligned}$$

*Then  $z_\eta$  has a nontrivial, hyperinvariant subspace.*

*Proof.* Note that we may without loss of generality take  $d < a$ . The conditions of the theorem are illustrated in Figure 4.

By Lemma 5.7, if  $0 < \theta < d - c$  and  $n \in \mathbf{N}$ , then  $s_\theta(z_\eta^n) \geq s_{\frac{\theta}{d-c}}(z_{\eta'}^n)$ , where  $\eta'$  is  $r(d - c)$  times the Lebesgue measure supported on  $\{(x, y) \mid 0 \leq x \leq y \leq 1\}$ . Therefore, cf. Examples 4.3(iii),  $z_{\eta'}$  is a  $\text{DT}(\delta_0, \sqrt{r(d - c)})$ -operator, i.e. is  $\sqrt{r(d - c)}$  times a  $\text{DT}(\delta_0, 1)$ -operator  $T$ . By Śniady's result [18] on  $*$ -moments of  $T$ , it follows

that  $(T^*)^n T^n$  and  $(\frac{1}{n} T^* T)^n$  have the same  $*$ -moments. Hence, for any  $n \in \mathbf{N}$  and  $0 < \sigma < 1$ ,

$$s_\sigma(T^n) = s_\sigma((T^*)^n T^n)^{1/2} = n^{-n/2} s_\sigma((T^* T)^n)^{1/2} = n^{-n/2} s_\sigma(T^* T)^{n/2} = n^{-n/2} s_\sigma(T)^n.$$

Hence,

$$s_\theta(z_\eta^n) \geq s_{\frac{\theta}{d-c}}(z_\eta^n) = \left( \frac{r(d-c)}{n} \right)^{n/2} s_{\frac{\theta}{d-c}}(T)^n.$$

By [5], the operator  $T$  has trivial kernel (in fact, the distribution of  $T^* T$  was explicitly determined there). Fixing any  $\theta \in (0, d-c)$ , we get  $s_{\frac{\theta}{d-c}}(T) \neq 0$ , and

$$s_\theta(z_\eta^n) \geq \left( \frac{\alpha}{n} \right)^{n/2}$$

for some  $\alpha > 0$ .

We may apply Lemma 5.6 to  $z_\eta$ . Let  $K$  be as in that lemma, and choose  $\gamma$  sufficiently close to 1 so that  $K(1-\gamma) \leq \frac{\alpha}{e}$ . Then choosing  $\mu_n$  as in (33) and using Stirling's formula for  $n!$ , we have

$$\limsup_{n \rightarrow \infty} \frac{\mu_n}{s_\theta(z_\eta^n)^2} \leq \limsup_{n \rightarrow \infty} \frac{n^n}{n!} \left( \frac{K(1-\gamma)}{\alpha} \right)^n = \limsup_{n \rightarrow \infty} \frac{1}{c_n \sqrt{n}} \left( \frac{eK(1-\gamma)}{\alpha} \right)^n = 0,$$

where  $c_n$  converges to a strictly positive number. Therefore, Theorem 3.1 applies, with  $p = 0$ , and yields a nontrivial hyperinvariant subspace for  $z_\eta$ .  $\square$

## 6. $L^\infty([0, 1])$ -CIRCULAR OPERATORS IN FREE GROUP FACTORS

In this section, we construct an  $L^\infty([0, 1])$ -circular operator  $z_\eta$ , as in Definition 5.2, inside of a free group factor, when  $\eta$  is assumed to be absolutely continuous with respect to Lebesgue measure. This construction parallels what was done in [5, §4] for the quasinilpotent DT-operator.

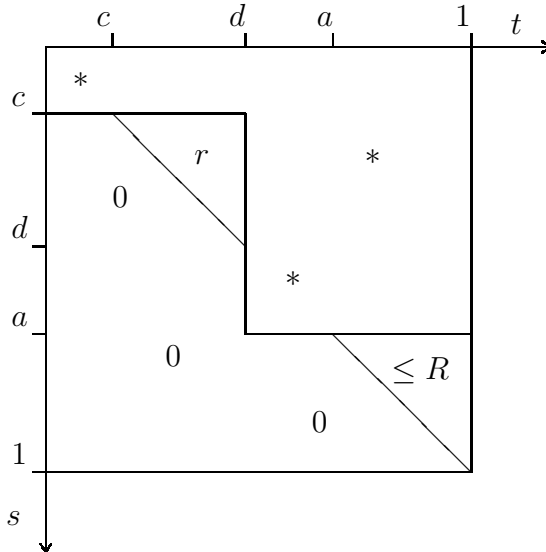


FIGURE 4. Conditions on  $\eta$  from Theorem 5.8.

As in the previous section,  $\mathcal{D}$  will denote  $L^\infty([0, 1])$  with trace  $\tau$  given by integration with respect to Lebesgue measure,  $(\mathcal{M}, \mathcal{E})$  will be a  $\mathcal{D}$ -valued  $W^*$ -noncommutative probability space such that  $\tau \circ \mathcal{E}$  is a normal, faithful, tracial state on  $\mathcal{M}$ , and  $z_\eta$  will be a  $\mathcal{D}$ -circular operator in  $(\mathcal{M}, \mathcal{E})$  with covariance  $(\alpha_\eta, \beta_\eta)$ , described by equations (31). We write  $\tau$  also for the trace  $\tau \circ \mathcal{E}$ . Let  $H \in L^1([0, 1])$  be the Radon–Nikodym derivative of  $\eta$  with respect to Lebesgue measure. Then

$$\begin{aligned} (\alpha_\eta f)(t) &= \int_0^1 H(s, t) f(s) ds \\ (\beta_\eta f)(t) &= \int_0^1 H(t, u) f(u) du. \end{aligned} \tag{38}$$

Moreover, the push-forward measures  $(\pi_i)_* \eta$  of  $\eta$  under the coordinate projections  $\pi_1$  and  $\pi_2$  are absolutely continuous with respect to Lebesgue measure and have Radon–Nikodym derivatives equal to the *coordinate expectations*  $CE_1(H)$  and  $CE_2(H)$ , respectively, given by

$$\begin{aligned} CE_1(H)(x) &= \int_0^1 H(x, y) dy \\ CE_2(H)(y) &= \int_0^1 H(x, y) dx. \end{aligned} \tag{39}$$

Thus, the assumption on  $\eta$  from Definition 5.2 is that  $CE_1(H)$  and  $CE_2(H)$  are essentially bounded, and then from Proposition 4.14 and equation (29) of Lemma 5.1, we have

$$\|z_\eta\| \leq 2 \max(\|CE_1(H)\|_\infty, \|CE_2(H)\|_\infty)^{1/2}. \tag{40}$$

*Definition 6.1.* Let  $w \in L^\infty([0, 1]^2)$ . We will say  $w$  is in *regular block form* if  $w$  is constant on all blocks in the regular  $n \times n$  lattice superimposed on  $[0, 1]^2$ , for some  $n$ , i.e. if there are  $n \in \mathbf{N}$  and  $w_{ij} \in \mathbf{C}$ ,  $(1 \leq i, j \leq n)$ , such that  $w(s, t) = w_{ij}$  whenever  $\frac{i-1}{n} \leq s < \frac{i}{n}$  and  $\frac{j-1}{n} \leq t < \frac{j}{n}$ , for all integers  $1 \leq i, j \leq n$ . For specificity, we may then say that  $w$  is in  $n \times n$  regular block form. Let  $a \in \mathcal{M}$ . Then we set

$$M(w, a) = \sum_{i,j=1}^n w_{ij} p_i a p_j \in \mathcal{M},$$

where  $p_i = \chi_{[\frac{i-1}{n}, \frac{i}{n}]} \in \mathcal{D}$ .

The following properties are straightforward.

- Lemma 6.2.** (a)  $M(w, a)$  is independent of the choice of  $n$  so long as  $w$  is in  $n \times n$  regular block form.  
 (b)  $M(w, \zeta a_1 + a_2) = \zeta M(w, a_1) + M(w, a_2)$  for  $a_1, a_2 \in \mathcal{M}$  and  $\zeta \in \mathbf{C}$ .  
 (c) If  $w^{(1)}, w^{(2)} \in L^\infty([0, 1]^2)$  are both in regular block form, then there is  $n \in \mathbf{N}$  such that both are in  $n \times n$  regular block form; for  $\zeta \in \mathbf{C}$ , we then have

$$M(\zeta w^{(1)} + w^{(2)}, a) = \zeta M(w^{(1)}, a) + M(w^{(2)}, a).$$



For the rest of this section, we will suppose that  $z \in \mathcal{M}$  is (scalar) circular with respect to  $\tau$  and satisfies  $\tau(z) = 0$  and  $\tau(z^*z) = 1$  and that  $\mathcal{D}$  and  $z$  are  $*$ -free (over  $\mathbb{C}$ ) with respect to  $\tau$ . Therefore,  $W^*(\mathcal{D} \cup \{z\}) \cong L(\mathbf{F}_3)$ .

**Lemma 6.3.** *Let  $w \in L^\infty([0, 1]^2)$  be in regular block form, let  $H = |w|^2$  and let  $\eta$  be the Lebesgue-absolutely-continuous measure on  $[0, 1]^2$  whose Radon-Nikodym derivative is  $H$ . Then  $M(w, z)$  is  $\mathcal{D}$ -circular in  $(\mathcal{M}, \mathcal{E})$ , with covariance  $(\alpha_\eta, \beta_\eta)$ .*

*Proof.* For brevity, write  $a$  for  $M(w, z)$ . Suppose  $w$  is in  $n \times n$  regular block form with  $w_{ij}$  as in Definition 6.1. The equalities

$$\begin{aligned} \mathcal{E}(a^*fa) &= \alpha_\eta(f) \\ \mathcal{E}(afa^*) &= \beta_\eta(f) \\ \mathcal{E}(afa) &= \mathcal{E}(a^*fa^*) = \mathcal{E}(a) = \mathcal{E}(a^*) = 0 \end{aligned} \tag{41}$$

for  $f \in \mathcal{D}$ , with  $\alpha_\eta$  and  $\beta_\eta$  as in (38), are easily verified using freeness. For example,

$$\begin{aligned} \mathcal{E}(a^*fa) &= \sum_{i,j,k=1}^n \overline{w_{ji}} w_{jk} \mathcal{E}(p_i z^* p_j f p_j z p_k) = \sum_{k=1}^n p_k \sum_{j=1}^n |w_{jk}|^2 \mathcal{E}(z^* p_j f z) \\ &= \sum_{k=1}^n p_k \sum_{j=1}^n |w_{jk}|^2 \tau(p_j f) = \sum_{k=1}^n p_k \sum_{j=1}^n |w_{jk}|^2 \int_{(j-1)/n}^{j/n} f(s) ds = \alpha_\eta(f). \end{aligned}$$

Suppose  $z_1, z_2, \dots \in \mathcal{M}$  are (scalar) circular elements such that  $\tau(z_j) = 0$  and  $\tau(z_j^* z_j) = 1$  and the family  $\mathcal{D}, \{z_1\}, \{z_2\}, \dots$  is  $*$ -free with respect to  $\tau$ . Then the family  $(\text{alg}(\{z_j, z_j^*\} \cup \mathcal{D}))_{j=1}^\infty$  is free over  $\mathcal{D}$  with respect to  $\mathcal{E}$ . Hence, by [19, Thm. 4.2.4], letting

$$a_k = \frac{1}{\sqrt{k}} (M(w, z_1) + \dots + M(w, z_k))$$

the pairs  $(a_k, a_k^*)$  converge in  $\mathcal{D}$ -valued moments with respect to  $\mathcal{E}$  to  $\mathcal{D}$ -Gaussian elements with covariance given by (41) as  $k \rightarrow \infty$ . In other words,  $a_k$  converges in  $\mathcal{D}$ -valued  $*$ -moments to a  $\mathcal{D}$ -circular element with covariance  $(\alpha_\eta, \beta_\eta)$ . However, by Lemma 6.2,

$$a_k = M\left(w, \frac{z_1 + \dots + z_k}{\sqrt{k}}\right).$$

But  $z' := \frac{z_1 + \dots + z_k}{\sqrt{k}}$  is a (scalar) circular element with  $\tau(z') = 0$  and  $\tau((z')^* z') = 1$  and with  $\mathcal{D}$  and  $z'$   $*$ -free. Thus,  $a_k$  has the same  $\mathcal{D}$ -valued  $*$ -moments as  $a$  itself, so  $a$  is  $\mathcal{D}$ -circular with covariance  $(\alpha_\eta, \beta_\eta)$ .  $\square$

**Lemma 6.4.** *Let  $H \in L^1([0, 1])$ ,  $H \geq 0$  and assume the coordinate expectations  $CE_1(H)$  and  $CE_2(H)$  as in (40) are essentially bounded. Let  $\eta$  be the Lebesgue-absolutely-continuous measure on  $[0, 1]^2$  whose Radon-Nikodym derivative is  $H$ . Let  $w = \sqrt{H}$ . Suppose there is a sequence  $(w^{(n)})_{n=1}^\infty$  in  $L^\infty([0, 1]^2)$  such that*

- (i) *for each  $n$ ,  $w^{(n)}$  is in regular block form,*
- (ii)  $\lim_{n \rightarrow \infty} \|w - w^{(n)}\|_{L^2([0, 1]^2)} = 0$ ,
- (iii) *letting  $H^{(n)} = |w^{(n)}|^2$ , both  $\|CE_1(H^{(n)})\|_\infty$  and  $\|CE_2(H^{(n)})\|_\infty$  remain bounded as  $n \rightarrow \infty$ .*

Let  $a_n = M(w^{(n)}, z)$ , with  $z$  as in Lemma 6.3. Then  $a_n$  converges in strong-operator topology (in the representation of  $\mathcal{M}$  on  $L^2(\mathcal{M}, \tau)$ ) to an element of  $\mathcal{M}$  which is a  $\mathcal{D}$ -circular operator with covariance  $(\alpha_\eta, \beta_\eta)$ .

*Proof.* By Lemma 6.3 and (40),

$$\|a_n\| \leq 2 \max(\|CE_1(H^{(n)})\|_\infty, \|CE_2(H^{(n)})\|_\infty)^{1/2},$$

so  $\|a_n\|$  remains bounded as  $n \rightarrow \infty$ . From Lemma 6.2, we have

$$a_n - a_m = M(w^{(n)} - w^{(m)}, z).$$

By Lemma 6.3,  $a_n - a_m$  is  $\mathcal{D}$ -circular with covariance corresponding to the measure on  $[0, 1]^2$  whose Radon–Nikodym derivative is  $|w^{(n)} - w^{(m)}|^2$ . Thus, from Lemma 5.3 we obtain

$$\|a_n - a_m\|_2 = \|w^{(n)} - w^{(m)}\|_{L^2([0,1]^2)}.$$

Therefore,  $a_n$  is Cauchy in  $L^2(\mathcal{M}, \tau)$ . Since  $\|a_n\|$  remains bounded, it follows that  $a_n$  converges in strong-operator topology to an element  $a$  of  $\mathcal{M}$ .

It follows, too, that the  $\mathcal{D}$ -valued  $*$ -moments of  $a_n$  converge in strong-operator topology to those of  $a$  as  $n \rightarrow \infty$ , in the sense that

$$\text{s.o.t-}\lim_{n \rightarrow \infty} \mathcal{E}(d_0 a_n^{s(1)} d_1 \cdots a_n^{s(k)} d_k) = \mathcal{E}(d_0 a^{s(1)} d_1 \cdots a^{s(k)} d_k)$$

for all  $k \in \mathbf{N}$ ,  $d_0, \dots, d_k \in \mathcal{D}$  and  $s(1), \dots, s(k) \in \{1, *\}$ . Therefore, the  $\mathcal{D}$ -valued free cumulants of  $a_n$  converge to those of  $a$  in strong-operator topology as  $n \rightarrow \infty$ . Let  $\eta_n$  be the Lebesgue absolutely continuous measure on  $[0, 1]^2$  whose Radon–Nikodym derivative is  $H^{(n)}$ . By Lemma 6.3,  $a_n$  is  $\mathcal{D}$ -circular with covariance  $(\alpha_{\eta_n}, \beta_{\eta_n})$ , and it follows that  $a$  is  $\mathcal{D}$ -circular with covariance  $(\alpha, \beta)$ , where for  $f \in \mathcal{D}$ ,

$$\begin{aligned} \alpha(f) &= \text{s.o.t-}\lim_{n \rightarrow \infty} \alpha_{\eta_n}(f) \\ \beta(f) &= \text{s.o.t-}\lim_{n \rightarrow \infty} \beta_{\eta_n}(f). \end{aligned}$$

From (38), we have

$$\begin{aligned} (\alpha_{\eta_n} f)(t) &= \int_0^1 |w^{(n)}(s, t)|^2 f(s) ds \\ (\beta_{\eta_n} f)(t) &= \int_0^1 |w^{(n)}(t, u)|^2 f(u) du. \end{aligned}$$

But

$$\begin{aligned} \|\alpha_{\eta_n}(f) - \alpha_\eta(f)\|_{L^1([0,1])} &= \int_0^1 |(\alpha_{\eta_n}(f) - \alpha_\eta(f))(t)| dt \\ &= \int_0^1 \left| \int_0^1 (|w^{(n)}(s, t)|^2 - |w(s, t)|^2) f(s) ds \right| dt \\ &\leq \|(|w^{(n)}|^2 - |w|^2)\|_{L^1([0,1]^2)} \|f\|_\infty \\ &\leq \|(|w^{(n)}| - |w|)\|_{L^2([0,1]^2)} (\|w^{(n)}\|_{L^2([0,1]^2)} + \|w\|_{L^2([0,1]^2)}) \|f\|_\infty, \end{aligned}$$

and  $\|w^{(n)}\|_{L^2([0,1]^2)}$  remains bounded as  $n \rightarrow \infty$ , while  $\|(|w^{(n)}| - |w|)\|_{L^2([0,1]^2)} \leq \|w^{(n)} - w\|_{L^2([0,1]^2)}$  tends to zero. Therefore,  $\alpha = \alpha_\eta$ . Similarly, we find  $\beta = \beta_\eta$ .  $\square$

**Theorem 6.5.** *Let  $H \in L^1([0,1]^2)$  have essentially bounded coordinate expectations  $CE_1(H)$  and  $CE_2(H)$ . Let  $\eta$  be the Lebesgue absolutely continuous measure on  $[0,1]^2$  whose Radon–Nikodym derivative is  $H$ . Then there is a  $\mathcal{D}$ -circular operator  $z_\eta$  with covariance  $(\alpha_\eta, \beta_\eta)$  in  $W^*(\mathcal{D} \cup \{z\}) \cong L(\mathbf{F}_3)$ .*

*Proof.* Let  $w = \sqrt{H}$ . By Lemma 6.4, it will suffice to find a sequence  $(w^{(n)})_{n=1}^\infty$  satisfying hypotheses (i)–(iii) listed there. For integers  $1 \leq i, j \leq n$ , let

$$w_{ij}^{(n)} = n^2 \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} w(x, y) dy dx$$

and let  $w^{(n)}(s, t) = w_{ij}^{(n)}$  whenever  $(s, t) \in [\frac{i-1}{n}, \frac{i}{n}) \times [\frac{j-1}{n}, \frac{j}{n})$ . Then  $w^{(n)}$  is in  $n \times n$  regular block form; i.e. (i) holds. Let us show

$$\lim_{n \rightarrow \infty} \|w^{(n)} - w\|_{L^2([0,1]^2)} = 0. \quad (42)$$

Let  $\epsilon > 0$ . There is a continuous function  $\tilde{w} : [0, 1]^2 \rightarrow [0, \infty)$  such that  $\|w - \tilde{w}\|_{L^2} < \epsilon$ . Let

$$\tilde{w}_{ij}^{(n)} = n^2 \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} \tilde{w}(x, y) dy dx$$

and let  $\tilde{w}^{(n)}(s, t) = \tilde{w}_{ij}^{(n)}$  whenever  $(s, t) \in [\frac{i-1}{n}, \frac{i}{n}) \times [\frac{j-1}{n}, \frac{j}{n})$ . Then

$$\|w^{(n)} - \tilde{w}^{(n)}\|_{L^2}^2 = n^{-2} \sum_{i,j=1}^n |w_{ij}^{(n)} - \tilde{w}_{ij}^{(n)}|^2.$$

But

$$\begin{aligned} |w_{ij}^{(n)} - \tilde{w}_{ij}^{(n)}| &\leq \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} |w(x, y) - \tilde{w}(x, y)| (n^2) dy dx \\ &\leq \left( \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} |w(x, y) - \tilde{w}(x, y)|^2 (n^2) dy dx \right)^{1/2} \\ &= n \left( \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} |w(x, y) - \tilde{w}(x, y)|^2 dy dx \right)^{1/2}, \end{aligned}$$

where the second inequality is because  $(n^2) dy dx$  is a probability measure on  $[\frac{i-1}{n}, \frac{i}{n}) \times [\frac{j-1}{n}, \frac{j}{n})$ . Therefore,

$$\|w^{(n)} - \tilde{w}^{(n)}\|_{L^2}^2 \leq \sum_{i,j=1}^n \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} |w(x, y) - \tilde{w}(x, y)|^2 dy dx = \|w - \tilde{w}\|_{L^2}^2 < \epsilon^2.$$

By uniform continuity of  $\tilde{w}$ ,  $\lim_{n \rightarrow \infty} \|\tilde{w} - \tilde{w}^{(n)}\|_{L^2} = 0$ , and using the triangle inequality, we get  $\|w - w^{(n)}\|_{L^2} < 3\epsilon$  for  $n$  sufficiently large. This proves (42), namely that hypothesis (ii) holds.

Finally, for (iii), letting  $H^{(n)} = |w^{(n)}|^2$ , we wish to show that  $\|CE_1(H^{(n)})\|_\infty$  and  $\|CE_2(H^{(n)})\|_\infty$  remain bounded as  $n \rightarrow \infty$ . We have, for  $x \in [\frac{i-1}{n}, \frac{i}{n}]$ ,

$$(CE_1(H^{(n)}))(x) = \int_0^1 |w^{(n)}(x, y)|^2 dy = \frac{1}{n} \sum_{j=1}^n |w_{ij}^{(n)}|^2.$$

But

$$\begin{aligned} |w_{ij}^{(n)}| &= \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} w(x, y) (n^2) dy dx \\ &\leq \left( \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} |w(x, y)|^2 (n^2) dy dx \right)^{1/2} \\ &= n \left( \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} |w(x, y)|^2 dy dx \right)^{1/2} \end{aligned}$$

so

$$\begin{aligned} (CE_1(H^{(n)}))(x) &\leq n \sum_{j=1}^n \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} |w(x, y)|^2 dy dx \\ &= n \int_{(i-1)/n}^{i/n} \int_0^1 |w(x, y)|^2 dy dx \\ &= n \int_{(i-1)/n}^{i/n} (CE_1(H))(x) dx \leq \|CE_1(H)\|_\infty \end{aligned}$$

and  $\|CE_1(H^{(n)})\|_\infty \leq \|CE_1(H)\|_\infty$ . Similarly, we get  $\|CE_2(H^{(n)})\|_\infty \leq \|CE_2(H)\|_\infty$ , and (iii) holds.  $\square$

## 7. SOME QUASINILPOTENT $L^\infty([0, 1])$ -CIRCULAR OPERATORS

As mentioned in the introduction, the existence of nontrivial hyperinvariant subspaces is presently of special interest for quasinilpotent operators in  $\Pi_1$ -factors. In this section we give sufficient conditions for an  $L^\infty([0, 1])$ -circular operator to be quasinilpotent.

**Lemma 7.1.** *Let  $z_\eta$  be an  $L^\infty([0, 1])$ -circular operator as in Definition 5.2 and suppose  $\eta$  is supported on the set*

$$\{(s, t) \mid 0 \leq s \leq t \leq 1\}$$

*Let  $\epsilon \in (0, 1)$ , let  $\eta_\epsilon$  be the restriction of  $\eta$  to*

$$\{(s, t) \mid 0 \leq s \leq t \leq 1, t - s \leq \epsilon\}$$

*and let  $z_{\eta_\epsilon}$  be the corresponding  $L^\infty([0, 1])$ -circular operator. Then the spectral radius  $r(z_\eta)$  of  $z_\eta$  is bounded above by  $\|z_{\eta_\epsilon}\|$ .*

*Proof.* Let  $\eta'_\epsilon = \eta - \eta_\epsilon$ . Then by Proposition 4.5(ii),  $z_\eta$  has the same  $*$ -moments as  $w + w'$ , where  $w$  and  $w'$  are  $L^\infty([0, 1])$ -circular elements with covariances  $(\alpha_{\eta_\epsilon}, \beta_{\eta_\epsilon})$  and  $(\alpha_{\eta'_\epsilon}, \beta_{\eta'_\epsilon})$ , respectively, and where  $w$  and  $w'$  are  $*$ -free over  $L^\infty([0, 1])$ . Given  $r \in [0, 1]$ , since

$$\eta_\epsilon([r, 1] \times [0, r]) = 0 = \eta'_\epsilon([r - \epsilon, 1] \times [0, r]),$$

from Lemma 5.5, we have

$$w\chi_{[0, r]} = \chi_{[0, r]}w\chi_{[0, r]}, \quad w'\chi_{[0, r]} = \chi_{[0, r-\epsilon]}w'\chi_{[0, r]},$$

where of course  $\chi_{[0, r-\epsilon]} = 0$  if  $r \leq \epsilon$ . Letting  $p$  be the least integer such that  $p\epsilon \geq 1$ , we therefore have  $(w')^p = 0$  and, given integers  $k(0), k(1), \dots, k(p) \geq 0$ , we also have

$$w^{k(0)}w'w^{k(1)}w' \dots w^{k(p-1)}w'w^{k(p)} = 0. \quad (43)$$

Since  $\|z_\eta^n\| = \|(w + w')^n\|$ , by distributing  $(w + w')^n$  and using (43), for  $n \geq p$  we obtain

$$\|z_\eta^n\| \leq \sum_{n=0}^{p-1} \binom{n}{q} \|w\|^{n-q} \|w'\|^q \leq pn^p \max(\|w\|^n, \|w\|^{n-p-1}) \max(1, \|w'\|^{p-1}).$$

Therefore, the spectral radius  $r(z_\eta) = \lim_{n \rightarrow \infty} \|z_\eta^n\|^{1/n}$ , is bounded above by  $\|w\| = \|z_{\eta_\epsilon}\|$ .  $\square$

**Proposition 7.2.** *Let  $z_\eta$  be an  $L^\infty([0, 1])$ -circular operator as in Definition 5.2. Suppose  $\eta$  is supported on the set*

$$\{(s, t) \mid 0 \leq s \leq t \leq 1\}$$

*and for some  $\delta > 0$ , the restriction of  $\eta$  to*

$$\{(s, t) \mid 0 \leq s \leq t \leq 1, t - s \leq \delta\} \quad (44)$$

*is absolutely continuous with respect to Lebesgue measure and has bounded Radon-Nikodym derivative. Then  $z_\eta$  is quasinilpotent.*

*Proof.* For  $0 < \epsilon \leq \delta$ , let  $\eta_\epsilon$  be as in Lemma 7.1 and let  $H_\epsilon$  be the Radon-Nikodym derivative of  $\eta_\epsilon$  with respect to Lebesgue measure on  $[0, 1]^2$ . In this context, equations (39) and (40) become

$$\|z_{\eta_\epsilon}\| \leq 2 \max(\|CE_1(H_\epsilon)\|_\infty, \|CE_2(H_\epsilon)\|_\infty)^{1/2}, \quad (45)$$

where

$$CE_1(H_\epsilon) = \int_x^{\min(x+\epsilon, 1)} H_\epsilon(x, y) dy$$

$$CE_2(H_\epsilon) = \int_{\max(0, y-\epsilon)}^y H_\epsilon(x, y) dx.$$

We thus obtain

$$\|CE_1(H_\epsilon)\|_\infty, \|CE_2(H_\epsilon)\|_\infty \leq \epsilon \|H_\epsilon\|_\infty \leq \epsilon \|H_\delta\|_\infty.$$

Consequently, from (45),  $\|z_{\eta_\epsilon}\| \leq 2\|H_\delta\|_\infty^{1/2} \sqrt{\epsilon}$ . Letting  $\epsilon \rightarrow 0$  and applying Lemma 7.1 yields  $r(z_\eta) = 0$ .  $\square$

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